Departement Physik, Universität Basel
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Mechanik, Herbstsemester 2026

Blatt 8

Abgabe: 10.11.2026, 12:00H, **entweder auf adam in den entsprechenden Ordner, oder in das Fach im Treppenhaus 4. Stock!**
Tutor: Marcelo Janovitch, office 515 Phys. Chem.; m.janovitch@unibas.c

(1) **Angular velocity in the body system and Euler angles** (3 Punkte)
The angular velocity $\boldsymbol{\omega}$ in the body frame can be expressed in terms of the time-dependent Euler angles by projecting the Euler rotations onto the body axes. Show that

$$\boldsymbol{\omega} = \begin{pmatrix} \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \dot{\varphi}\cos\theta + \dot{\psi} \end{pmatrix} .$$

Hint: Use $\boldsymbol{\omega} = \dot{\varphi}\mathbf{e}_z + \dot{\theta}\mathbf{e}_{ON} + \dot{\psi}\mathbf{e}_3$ and express the unit vectors $\mathbf{e}_z$ and $\mathbf{e}_{ON}$ pointing in the direction of the z-axis and the axis ON in terms of the unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ pointing in the direction of the body-fixed axes.

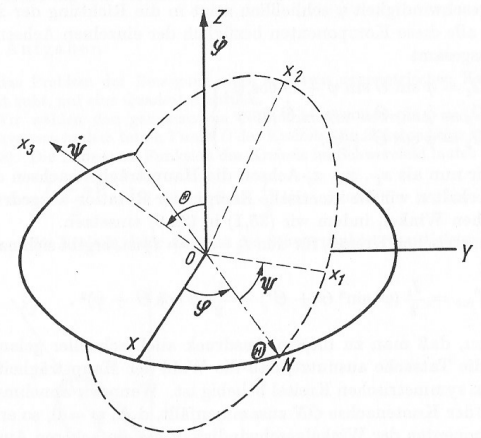


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(2) **Free symmetric top in the body frame and in the lab frame** (3 Punkte)
In the lecture we solved Euler's equation for a free (i.e., no torques) symmetric top ($I_1 = I_2 =: I \neq I_3$).

(a) Describe and sketch the motion of the angular momentum $\mathbf{L}$, the angular velocity $\boldsymbol{\omega}$, and the principal axis $\mathbf{e}_3$ as seen in the body frame. Distinguish the cases $I_3 > I$ and $I_3 < I$ (which case corresponds to a coin-like shape, which one to a carrot-like shape?).

(b) Now we want to analyze the motion in the lab frame. In terms of the (changing) principal axes $\mathbf{e}_i$, we have

$$\begin{aligned}\boldsymbol{\omega} =&(\omega_1\mathbf{e}_1 + \omega_2\mathbf{e}_2) + \omega_3\mathbf{e}_3\\ \mathbf{L} =&I(\omega_1\mathbf{e}_1 + \omega_2\mathbf{e}_2) + I_3\omega_3\mathbf{e}_3\ .\end{aligned}$$

Eliminate $(\omega_1\mathbf{e}_1+\omega_2\mathbf{e}_2)$ from these two equations and express $\boldsymbol{\omega}$ in terms of $\mathbf{L}$ and $\mathbf{e}_3$. Result: $\boldsymbol{\omega} = \frac{\mathbf{L}}{I} - \xi\mathbf{e}_3$ where $\xi = \omega_3\frac{I_3-I}{I}$.
Conclude that $\mathbf{L}$, ω, and $\mathbf{e}_3$ lie in a plane.
Since $\mathbf{e}_3$ is fixed in the body frame, its change in the lab frame comes only from rotation around $\boldsymbol{\omega}$, i.e., $\dfrac{d\mathbf{e}_3}{dt} = \boldsymbol{\omega}\times\mathbf{e}_3$. Conclude that $\mathbf{e}_3$ precesses around $\tilde{\boldsymbol{\omega}} = \mathbf{L}/I$ (that is constant in the lab frame - why?) with frequency $\tilde{\omega} = \frac{L}{I}$.
Describe and sketch the motion of $\mathbf{L}$, $\boldsymbol{\omega}$, and $\mathbf{e}_3$ as seen in the lab frame. Distinguish the cases $I_3 > I$ and $I_3 < I$.

(c) (Bonus points) We found that a person standing on the rotating body sees $\mathbf{L}$ (and $\boldsymbol{\omega}$) precess with frequency $\xi = \omega_3\frac{I_3-I}{I}$ around $\mathbf{e}_3$. A person in the lab frame sees $\mathbf{e}_3$ (and $\boldsymbol{\omega}$) precess with frequency $\frac{L}{I}$ around $\mathbf{L}$. Are these two facts compatible?

(3) **Symmetric top in a uniform gravitational field ("heavy top")** (4 Punkte)
In the lecture we discussed the motion of a symmetric top placed in a uniform gravitational field and fixed in one point by expressing the Lagrangian using Euler angles,

$$L(\phi,\theta,\psi,\dot\phi,\dot\theta,\dot\psi) = \frac{1}{2}I_1(\dot\phi^2\sin^2\theta + \dot\theta^2) + \frac{1}{2}I_3(\dot\phi\cos\theta + \dot\psi)^2 - mg\ell\cos\theta\ .$$

(a) Assume that the top rotates around the vertical axis of the lab system, $\theta = 0$. Show that in this case, $L_z = L_3$ where $L_z = p_\phi = \partial L/\partial\dot\phi$ is the angular momentum about the vertical axis and $L_3 = p_\psi = \partial L/\partial\dot\psi$ the angular moment about the body axis x_3.

(b) Plot the effective potential for a representative choice of parameters. Find the condition under which the vertical rotation is *stable* under small perturbations θ . Hint: expand the effective potential around $\theta = 0$.

(c) Bonus points: solve the equations of motion (for general θ) numerically. Confirm the nutation behavior discussed in the lecture and the stability condition discussed in (b).