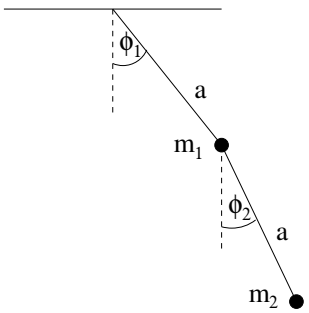
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Mechanik, Herbstsemester 2026

Blatt 7

Abgabe: 3.11.2026, 12:00H, **entweder auf adam in den entsprechenden Ordner, oder in das Fach im Treppenhaus 4. Stock!**
Tutor: Bethany Davies Zi.: 4.10; bethany.davies@unibas.ch

(1) **Double pendulum** (5 Punkte)
Consider a planar double pendulum (masses m_1, m_2, same length a) subject to a homogeneous gravitational field.



(a) Construct the Lagrangian by writing down the kinetic and potential energies of m_1 and m_2. Result: $L(\phi_1, \phi_2, \dot{\phi}_1, \dot{\phi}_2) = \frac{1}{2}(m_1 + m_2)a^2\dot{\phi}_1^2 + \frac{1}{2}m_2a^2\dot{\phi}_2^2$
$+ m_2a^2\cos(\phi_1 - \phi_2)\dot{\phi}_1\dot{\phi}_2 + (m_1 + m_2)ga\cos\phi_1 + m_2ga\cos\phi_2$.

(b) Expand the Lagrangian for small oscillations $\phi_1, \phi_2 \ll 1$. Keep terms up to quadratic order in ϕ_i, $\dot{\phi}_i$. Result: $L_{\text{small}} = \frac{1}{2}\sum_{i,j}^2 m_{ij}\dot{\phi}_i\dot{\phi}_j - \frac{1}{2}\sum_{i,j}^2 k_{ij}\phi_i\phi_j$
where $M = m_{ij} = a^2\begin{pmatrix} m_1 + m_2 & m_2 \\ m_2 & m_2 \end{pmatrix}$, $K = k_{ij} = ga\begin{pmatrix} m_1 + m_2 & 0 \\ 0 & m_2 \end{pmatrix}$.

(c) Write down the equations of motion.

(d) Calculate the eigenfrequencies and eigenmodes. Discuss and plot them as a function of m_2/m_1 and discuss their limiting behavior. What happens for

i. $m_2/m_1 \ll 1$? (don't simply set $m_2 = 0$)

ii. $m_2/m_1 \to \infty$?

(e) Calculate and discuss the time dependence of the motion for $m_1 = m_2$ and the initial condition $\phi_1 = 0$, $\phi_2 = \alpha \neq 0$, $\dot{\phi}_1 = \dot{\phi}_2 = 0$.

(2) **Bottle on the floor of a tram** (2 Punkte)
A bottle (modeled as a homogeneous circular cylinder of radius R and mass M) lies on the floor of a tram; its orientation is perpendicular to the tram's direction of motion. When the tram starts moving with acceleration a, the bottle will start to roll (we assume without sliding).

(a) Calculate the moment of inertia I along the symmetry axis. Result: $I = \frac{1}{2}MR^2$

(b) Determine the acceleration $\tilde{a}$ of the bottle relative to the passengers.

(3) **Inertia ellipsoid** (3 Punkte)
Consider a body with density $\rho(\mathbf{x})$ and inertia tensor $I_{\mu\nu}$. We define the set Γ of all points $\boldsymbol{\omega}$ for which $\boldsymbol{\omega}^T I \boldsymbol{\omega} = 1$.

(a) Show that Γ is an ellipsoid whose axes are parallel to the principle axes of the body and whose (semi-)axes lengths are $1/\sqrt{I_i}$ where I_i, $i = 1, 2, 3$ are the principle moments of inertia of the body.

(b) Assume that the body is invariant under a symmetry transformation R, $\rho(R\mathbf{x}) = \rho(\mathbf{x})$. Show that Γ is also invariant under R. Hint: $R^{\mathrm{T}}R = 1$.

(c) Show: if the body has a k-fold symmetry axis (i.e., is invariant under rotations by $2\pi/k$ about this axis) with $k \geq 3$, this axis is a principle axis and the body is a symmetric top. If the body has more than one such axis, it is a spherical top.

(4) **Bonus problem: Steiner's theorem (= Parallel axis theorem)**(2 Bonuspunkte)
Consider a rigid body of total mass M whose inertia tensor I^S_{ij} is given in the rigid-body coordinate system x_1, x_2, x_3 whose origin coincides with the center of mass S. If we shift this coordinate system by a vector $\mathbf{a}$, we obtain a new coordinate system x'_1, x'_2, x'_3 whose axes are parallel to the axes of x_1, x_2, x_3.

Show that the inertia tensor in the new coordinate system has the form

$$I_{ij} = I^S_{ij} + M(\mathbf{a}^2\delta_{ij} - a_i a_j) \, .$$