

Mechanik, Herbstsemester 2026

Blatt 4

Abgabe: 13.10.2026, 12:00H, **entweder auf adam in den entsprechenden Ordner, oder in das Fach im Treppenhaus 4. Stock!**

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(1) Circular cone revisited

(2 Punkte)

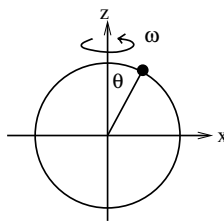
We would like to look at the particle moving on a circular cone (opening angle $2\alpha = \pi/2$) treated in problem 2 of Blatt 2 one more time: as we saw earlier, the Lagrangian in terms of the generalized (polar) coordinates r, φ is $L(r, \varphi, \dot{r}, \dot{\varphi}; t) = \frac{m}{2}(2\dot{r}^2 + r^2\dot{\varphi}^2) - mgr$. The variable φ is cyclic, hence $l_z = \partial L / \partial \dot{\varphi}$ is conserved. Also, since L does not depend explicitly on time, the energy $E = m\dot{r}^2 + l_z^2/(2mr^2) + mgr$ is conserved.

- Use E to write down a first-order differential equation for r and find the formal solution by separating the variables (you are not required to evaluate the integral).
- Discuss and sketch the allowed and forbidden regions by using the effective potential V_{eff} . Find the turning points of the radial coordinate where the energy is equal to V_{eff} and discuss your solution graphically. When does the equation have 0, 1, or 2 (physically relevant) solutions? Interpret the three different cases.

(2) Bead on a rotating ring

(4 Punkte)

A bead (mass m) is subject to a (homogeneous) gravitational force acting in the negative z -direction and moves without friction on a vertical circle (radius R) that rotates with angular velocity ω about the z -axis, see figure. It is convenient to use the angle θ as the generalized coordinate (why do we need only one?).



- Write down the kinetic and potential energies and express the Lagrangian $L = T - V$ in terms of the generalized coordinate.
Result: $L(\theta, \dot{\theta}; t) = \frac{m}{2}R^2\dot{\theta}^2 + \frac{m}{2}R^2\omega^2 \sin^2 \theta - mgR \cos \theta$.
- Conclude that the motion of the bead corresponds to a one-dimensional motion in an effective potential $U(\theta)$. Sketch and discuss U for the two cases $R\omega^2/g < 1$ and $R\omega^2/g > 1$. What are the allowed regions for a given total energy? Determine and discuss the stable equilibrium position(s) of the bead in both cases.
- Carefully sketch the phase portraits (i.e., $\dot{\theta}$ as a function of θ) in both cases (or plot them using a computer).

(3) **Numerical experiments** (4 Punkte + bonus points)

The goal of this problem is to study the motion of a particle in a variety of two-dimensional potentials. Start by plotting the potential $V(x, y)$ (either as a 3D- or contour plot). Calculate the force on the particle and write down the two Newton equations (for a particle of mass $m=1$).

Solve the differential equations numerically, preferably using Julia (a “skeleton” is provided in the notebook folder on adam), or else your favorite method.

Use the following initial conditions: $\dot{y}(0) = 0.5, 1$, and 2 , as well as always $x(0) = 1$, $y(0) = 0$, and $\dot{x}(0) = 0$. Plot the resulting trajectories $(x(t), y(t))$ in the xy -plane in the time interval $[t = 0, t_{\text{end}}]$ for the values of t_{end} given below.

- (a) $V(x, y) = -1/r$ where $r = \sqrt{x^2 + y^2}$ (Kepler problem of a particle in the gravitational field). What are the qualitatively different trajectories? ($t_{\text{end}} = 8$)
- (b) $V(x, y) = \ln(r)$ (another central potential. It corresponds to the Coulomb potential of a charged wire perpendicular to the xy -plane). Discuss the qualitative differences to (a). Why can you find a circular orbit in both cases? ($t_{\text{end}} = 20$)
- (c) $V(x, y) = x^2/2 + y^2$ (anisotropic two-dimensional harmonic oscillator). Why do the trajectories (so-called “Lissajous curves”) not close? ($t_{\text{end}} = 20$)
- (d) $V(x, y) = -(1 + \exp(10(\sin(x)^2 \sin(y)^2 - 1/2)))^{-1}$ (Chaotic motion in a quadratic lattice of scattering centers). Use the following two sets of initial conditions to test the influence of small changes of the initial conditions: $x(0) = 2$ or $x(0) = 2.1$, and always $y(0) = 0$, $\dot{x}(0) = 0$, $\dot{y}(0) = 0.5$. ($t_{\text{end}} = 40$)
- (e) Bonus points: find another interesting potential and discuss the resulting motion in a qualitative way!
- (f) Bonus points: consider a perturbed Kepler potential, $V(x, y) = -1/r + \beta/r^2$, where $\beta \ll 1$. Plot the trajectory for $\dot{y}(0) = 0.5$ and study the precession of the orbit as a function of β . The additional term looks very much like the centrifugal barrier term in the effective potential $V_{\text{eff}}(r)$. Why is it then that the additional force term causes a precession of the orbit, while an addition to the barrier, through a change in ℓ , does not?