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Mechanik, Herbstsemester 2026

Blatt 10

Abgabe: 24.11.2026, 12:00H, **entweder auf adam in den entsprechenden Ordner, oder in das Fach im Treppenhaus 4. Stock!**
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(1) **Properties of the Poisson bracket** (3 Punkte)
The Poisson bracket of two observables (i.e., functions on phase space) $f(\mathbf{q}, \mathbf{p}; t)$ and $g(\mathbf{q}, \mathbf{p}; t)$ is defined as

$$\{f, g\} \equiv \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)$$

where n is the number of degrees of freedom.

(a) Write down the equations of motion of the one-dimensional harmonic oscillator using the Poisson bracket.

(b) Show that the components of the angular momentum vector $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ fulfill $\{L_x, L_y\} = L_z$ (and cyclic permutations).

(2) **Phase portraits and Liouville's theorem** (5 Punkte)
Liouville's theorem states that the volume in phase space occupied by a collection of systems remains constant over time.

Consider the dimensionless Hamiltonian $H(q, p) = \frac{p^2}{2} + V(q)$, where $V(q)$ is given by

(a) $V(q) = \frac{q^2}{2}$ (harmonic oscillator).

(b) $V(q) = \frac{q^2}{2} + q^4$ (anharmonic oscillator).

(c) $V(q) = 1 - \cos(q)$ (pendulum; q corresponds to the deflection angle).

For each of these examples, draw phase portraits by plotting phase-space orbits $(q(t), p(t))$ for a number of equidistant energies. Discuss the difference between open and closed orbits in (c). What happens for the initial condition $q = 0$, $p = 2$ in (c)?

Consider now the rectangular phase-space volume $-\frac{1}{2} \leq q \leq \frac{1}{2}, 1 \leq p \leq 3)$ at $t = 0$. Analyze its time evolution qualitatively by carefully sketching (or numerically calculating) its shape for different times (at least for a time $t \approx 1$ and for a time $t \gg 1$).

(3) **Poincaré sections** (2 Punkte + 5 Bonuspunkte)

The goal of this problem is to calculate and discuss Poincaré sections.

(a) As a warm-up, consider the anisotropic two-dimensional harmonic oscillator

$$H = \frac{1}{2m}(p_1^2 + p_2^2) + \frac{1}{2}m\omega_1^2 q_1^2 + \frac{1}{2}m\omega_2^2 q_2^2$$

with $\omega_2 = \sqrt{2}\omega_1$ and the initial condition $q_1(0) = q_2(0) = 0$, $p_1(0) = p_2(0) = \sqrt{mE/2}$ where E is the energy. Calculate and sketch the points at which the trajectory crosses the q_2, p_2-plane with $q_1 = 0$, $p_1 > 0$. Repeat this for the points at which the trajectory crosses the q_1, p_1-plane with $q_2 = 0$, $p_2 > 0$.
Hint: you don't really need a computer to solve (a).

(b) Bonus point: what happens if the two frequencies are commensurate, $\omega_2 = n\omega_1$ with integer n?

Now to a more interesting case **that exhibits chaos**: the double pendulum. The Hamiltonian was given in problem 1(e) of Blatt 9, and we choose the parameters $m_1 = m_2 = 1$, $g = 1$, and $a = 1$.

(c) Use Julia (or your favorite numerical method) to calculate the points at which the trajectory crosses the q_2, p_2-plane with $q_1 = 0$, $p_1 > 0$. Use the initial conditions $p_1(0) = 0.1$, $p_2(0) = 1.4$, $q_1(0) = 0.1$, $q_2(0) = 0$ and integrate the equations of motion in the time interval $[0, 3000]$.
Hint: a Julia notebook that solves this task for the Hénon-Heiles potential that was discussed in the lecture can be found in the notebook folder on adam.

(d) Repeat the procedure for the q_1, p_1-plane with $q_2 = 0$, $p_2 > 0$ and the initial conditions $p_1(0) = 0$, $p_2(0) = 0$, $q_1(0) = 0.1$, $q_2(0) = 0.2$.

(e) We now assume that the energy E is given. Find initial conditions that are compatible with E. Calculate the Poincaré sections in the q_1, p_1-plane with $q_2 = 0$, $p_2 > 0$ for the following energies: $E = -2.9$, $E = 0$, and $E = 10$. Try other possibilities!