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Quantenmechanik, Herbstsemester 2025

Blatt 4

Abgabe: **Tuesday 14.10.25**, 12:00H (auf adam oder Treppenhaus 4. Stock)
Tutor: Tobias Kehrer, Zi. 4.48

(1) **Commutators of components of x and p** (2 Punkte)
Let F and G be analytic functions (i.e., they can be expanded in a power series) of $\mathbf{p}$ and $\mathbf{x}$, respectively.
Show (e.g., by induction) that

(a) $[x_j, F(\mathbf{p})] = i\hbar \dfrac{\partial F}{\partial p_j}(\mathbf{p})$

(b) $[p_j, G(\mathbf{x})] = -i\hbar \dfrac{\partial G}{\partial x_j}(\mathbf{x})$

(2) **Unequal time commutation relations** (2 Punkte)
Calculate the commutation relations $[\hat{x}_H(t), \hat{p}_H(t')]$ of the (one-dimensional) position and momentum operators in the Heisenberg picture at times t, t' for the following cases

(a) a particle acted on by a constant force

(b) a harmonic oscillator.

(3) **Time evolution of a free particle** (3 Punkte)
Consider a free particle in three dimensions, $\hat{H} = \hat{\mathbf{p}}^2/2m$. Calculate the commutator $[\hat{x}_{jH}(t), \hat{x}_{jH}(0)]$ of the position operator $\hat{\mathbf{x}}_H$ in the Heisenberg picture, here, $\hat{x}_{jH}$, $j = 1, 2, 3$ are the components of $\hat{\mathbf{x}}_H$.
Give a lower bound for $\Delta\hat{x}_{jH}(t)\,\Delta\hat{x}_{jH}(0)$ and interpret your result.
$\Delta A := \sqrt{\langle A^2\rangle - \langle A\rangle^2}$

(4) **Harmonic oscillator** (3 Punkte)
Consider a harmonic oscillator of mass m and angular frequency ω. At time $t = 0$, the state of this oscillator is given by $|\psi(0)\rangle = \sum_n c_n|n\rangle$ where the $|n\rangle$ are eigenstates with energies $\hbar\omega(n + 1/2)$ for $n \geq 0$.

(a) What is the probability W that a measurement of the oscillator's energy performed at an arbitrary time $t > 0$, will yield a result greater than $3\hbar\omega$? When $W = 0$, what are the non-zero coefficients c_n?

(b) Assume that only c_0 and c_2 are different from zero. Write the normalization condition for $|\psi(0)\rangle$ and the expectation value $\bar{E}$ of the energy in terms of c_0 and c_2. Calculate $|c_0|^2$ and $|c_2|^2$ if $\bar{E} = \hbar\omega$.

(c) If at time $t = 0$ the state of the oscillator is $|\psi(0)\rangle = \dfrac{1}{\sqrt{5}}(2|1\rangle + |2\rangle)$, calculate $|\psi(t)\rangle$ for $t > 0$ and the mean value $\langle x(t)\rangle$ of the position at t.

(5) **Peres-Horodecki criterion for separability** (4 Bonus-Punkte)
We consider a bipartite system (i.e., the total system consists of two subsystems) of two spin 1/2 particles and define the family of so-called Werner states by

$$\rho_W(p) = p|S\rangle\langle S| + \frac{1}{4}(1-p)\mathbb{1},$$

with $|S\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ and $0 \le p \le 1$.
For mixed states, we define "entangled" as follows: A state is entangled if it is not separable, i.e., *cannot* be written as a convex combination of product states

$$\rho = \sum_j \lambda_j \rho_j^{(1)} \otimes \rho_j^{(2)},$$

where $\rho_j^{(1)}$, $\rho_j^{(2)}$ are density operators of the two subsystems and $\lambda_j \ge 0$ such that $\sum_j \lambda_j = 1$.

(a) For $p = 0$ and $p = 1$ decompose $\rho_W(p)$ into a convex combination of product states or prove that no such decomposition exists.

(b) Show that if a state ρ is separable, then its partial transpose is positive semidefinite (Peres-Horodecki criterion).
Hint: For a state ρ defined on a Hilbert space $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$ the partial transpose (with respect to subsystem 2) is defined as $\tilde{\rho} = (\mathbb{1}^{(1)} \otimes T^{(2)})(\rho)$, where $T^{(2)}$ is the transposition map in $\mathcal{H}^{(2)}$.

(c) Use (b) to check for which values of p the state $\rho_W(p)$ is guaranteed to be entangled.