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Quantenmechanik, Herbstsemester 2025

Blatt 3

Abgabe: 07.10.25, 12:00H (auf adam oder Treppenhaus 4. Stock)
Tutor: Tobias Nadolny, Zi.: 4.48

(1) **State determination** (2 Punkte)

Measurements on a system of two spin 1/2 particles yield the following expectation values:

$$\left\langle S_z^{(1)} \right\rangle = \left\langle S_z^{(2)} \right\rangle = 0 \quad \text{and} \quad \left\langle S_z^{(1)} \otimes S_z^{(2)} \right\rangle = -\frac{\hbar^2}{4},$$

where $S_z = \frac{\hbar}{2}\sigma_z$.

(a) Construct a *pure* state consistent with the given data, or prove that none exists.

(b) Construct a *mixed* state consistent with the given data, or prove that none exists.

(2) **Reduced density operator** (3 Punkte)

Consider a system of two spins 1/2 in the state $|\psi_\alpha\rangle = \cos(\alpha)|\downarrow\downarrow\rangle + i\sin(\alpha)|\uparrow\uparrow\rangle$.

(a) Write down the density operator ρ that describes this system.

(b) Calculate the reduced density operator $\rho^{(1)}$ of subsystem 1 (i.e., the first spin). Does it represent a pure or a mixed state? What is the expectation value of $S_z^{(1)}$ and $S_x^{(1)}$ if only the first spin is measured. Interpret your results.

(c) For a joint measurement of $S_x^{(1)}$ and $S_x^{(2)}$ of both spins, calculate the probability to measure $\hbar/2$ for spin 1 and $\hbar/2$ for spin 2.

(3) **Time evolution** (3 Punkte)

Consider a spin 1/2 particle. At time $t = 0$, the system is prepared in the state

$$|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle_z - e^{i\phi}|\downarrow\rangle_z\Big) \,.$$

The Hamiltonian of the system is given by $H = -g\frac{\mu_B}{\hbar}\mathbf{B}\cdot\mathbf{S}$, with $\mathbf{B} = B_0\mathbf{e}_z$ and $\mathbf{S} = \frac{\hbar}{2}\left(\sigma_x, \sigma_y, \sigma_z\right)$.

(a) Write down a formal expression for the time evolution operator $U(t)$ and evaluate it explicitly. Calculate $|\psi(t)\rangle$ for $t > 0$.
Hint: One possibility is to show that $\exp\left(i\alpha\sigma_z\right) = \mathbb{1}\cos\alpha + i\sigma_z\sin\alpha$.

(b) What is the probability that a measurement of S_x done at time $t > 0$ gives $-\hbar/2$? What is the probability that a measurement of S_z done at time $t > 0$ gives $+\hbar/2$?

(c) Calculate $\langle S_x(t)\rangle$, $\langle S_y(t)\rangle$, and $\langle S_z(t)\rangle$. Interpret your result.

(4) **Quantum speed limit** (2 Punkte + 2 Bonuspunkte)

We consider a system described by the Hamiltonian H with eigenenergies E_n and eigenstates $|n\rangle$, i.e., $H|n\rangle = E_n|n\rangle$. Assume that the system is initially prepared in an arbitrary state $|\psi_0\rangle$. We want to show that there exists a fundamental lower bound on the time it takes the system to evolve into a state that is orthogonal to $|\psi_0\rangle$.

(a) Give an expression for $|\psi(t)\rangle$ using the initial condition $|\psi(t=0)\rangle = |\psi_0\rangle$.

(b) Now consider $S(t) := \langle\psi_0|\psi(t)\rangle$. We want to find the smallest value $t_{\min}$ of t such that $S(t_{\min}) = 0$. Write down an expression for $\operatorname{Re} S(t)$ and use the trigonometric inequality $\cos x \geq 1 - \frac{2}{\pi}(x + \sin(x))$ valid for $x \geq 0$ to show that

$$t_{\min} = \frac{\pi\hbar}{2E} \tag{1}$$

where $E = \langle\psi_0|H|\psi_0\rangle$ is the expectation value of H.

(c) Interpret your result.

(d) Bonus points: Consider a 2-level system and show that the bound (1) is achievable.