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Quantenmechanik, Herbstsemester 2025

Blatt 12 (Zusatzpunkte)

Abgabe: 9.12.25, 12:00H (Treppenhaus 4. Stock)

Schriftlicher Test: Dienstag, 16. Dezember 2025, 10.15 - 12 Uhr
Hilfsmittel: Ein beidseitig **hand**beschriebenes A4 Blatt.

(1) **Distinguishable (labeled) particles, fermions, bosons** (5 Bonuspunkte)

(a) Two identical bosons are found to be in states $|\phi\rangle$ and $|\psi\rangle$. Write down the normalized state vector describing the system when $\langle\phi|\psi\rangle \neq 0$.

(b) When an energy measurement is made on a system of three bosons in a box, the n values obtained were 3, 3, and 4. Write down a symmetrized, normalized state vector.

(c) Consider three particles each of which can be in states ϕ_a, ϕ_b, and ϕ_c. Show that the total number of allowed, distinct configurations for this system is

i. 27 if they are labeled
ii. 10 if they are bosons
iii. 1 if they are fermions

(2) **Toy model for bunching/antibunching** (5 Bonuspunkte)
Consider two particles in two orthonormal states $\psi_a(x_1)$, $\psi_b(x_2)$.

(a) Write down the 2-particle wave function for distinguishable particles, for indistinguishable bosons, and for indistinguishable fermions.

(b) Calculate the expectation value $\langle(\Delta x)^2\rangle := \langle(x_1 - x_2)^2\rangle$ of the square of the distance between the two particles for all three cases and interpret your result.

(c) Do this calculation for two particles in states ψ_n and ψ_m in an infinite square well.

Weitere Aufgaben für Unentwegte.
Abgabe bis 23.1.2026, Besprechung nach Vereinbarung

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(3) **Attractive spherical δ-function shell** (10 Punkte)

Problem 3 on Blatt 11 studied the scattering amplitude and scattering phase shift of a repulsive spherical δ-function shell. The goal of the present problem is to analyze the very interesting case of an *attractive* shell that exhibits both bound states and so-called scattering resonances.

Investigate the bound states of the three-dimensional δ-shell potential

$$V(r) = -\alpha\delta(r-a)\,.$$

It is useful to introduce the dimensionless variables $y := r/a$, $\xi := ka$, and $\beta := 2ma\alpha/\hbar^2$. It turns out that there is at most one bound state for each l.

(a) Determine the s-wave function. Show that a bound state exists only for $\beta > 1$.

(b) Show that there is at most one bound state corresponding to each l.

(c) Show for general l that the minimum strength of the potential for the existence of a bound state is $\beta = 2l + 1$.

(d) Calculate the scattering phases $\delta_l(k)$.

(e) Give the scattering cross section for s-waves.

(f) Determine the condition for the maxima of the s-wave scattering cross section.

(g) From here on, assume $\beta \gg \pi$. Determine the maxima for $ka \ll \beta$.

(h) Show that there are sharp and broad resonances. Show that the Breit-Wigner formula $\sigma \sim \Gamma^2/[(E-E_R)^2+\Gamma^2]$ holds near the sharp resonances.

(i) Determine the poles of $e^{2i\delta_l} - 1$ on the negative real E-axis and interpret them.

(4) **Fractional quantum Hall effect** (5 Punkte)

The Laughlin wave function

$$\psi(z_1, z_2, \ldots, z_N) = A\left[\prod_{j<k}^{N}(z_j - z_k)^q\right]\exp[-\frac{1}{2}\sum_{k}^{N}|z_k|^2]$$

is an approximate description of the ground state of N interacting electrons confined to two dimensions in a perpendicular magnetic field B. Here, q is a positive integer, and $z_j := \sqrt{\dfrac{eB}{2\hbar c}}(x_j + iy_j)$. Spin is not an issue here: in the ground state, all the electrons have spin down with respect to the direction of $\mathbf{B}$, and that is a trivially symmetric configuration.

(a) Show that ψ has the proper antisymmetry for fermions.

(b) For $q = 1$, ψ describes noninteracting particles (i.e., can be written as a single Slater determinant). Check this explicitly for $N = 3$. What single-particle states are occupied in this case?

(c) For $q > 1$, ψ *cannot* be written as a single Slater determinant and describes interacting particles. It can, however, be written as a sum of Slater determinants. Show that, for $q = 3$ and $N = 2$, ψ can be written as a sum of two Slater determinants.

(5) **Shifted creation/annihilation operators** (5 Punkte)
Study the shift from the (bosonic or fermionic) operators $\hat{a}$, $\hat{a}^\dagger$ to new operators $\hat{a}' = \hat{a} + \alpha$, $\hat{a}'^\dagger = \hat{a}^\dagger + \alpha^*$; here α is a complex number. Is this transformation unitary, i.e., does it preserve the fermionic/bosonic commutation relations? If so, give an explicit expression for the unitary operator.
Hint: $e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + \frac{1}{1!}[\hat{A}, \hat{B}] + \frac{1}{2!}[\hat{A}, [\hat{A}, \hat{B}]] + \dots$

(6) **Tight-binding model in second quantization** (5 Punkte)
A major part of solid-state physics deals with electrons in a periodic potential. As a simplified model we consider fermionic particles moving on a cubic lattice (lattice constant a). The kinetic energy is assumed to have tight-binding form

$$H = -t \sum_{<i,j>\sigma} \left[c^\dagger_{i\sigma} c_{j\sigma} + c^\dagger_{j\sigma} c_{i\sigma} \right] ,$$

here, $\sum_{<i,j>}$ is the sum over all nearest neighbors (such that each bond appears only once) and $\sum_\sigma$ is the sum over the two spin directions.

(a) Determine the band structure $\epsilon(\mathbf{k})$ for a d-dimensional cubic lattice ($d = 1, 2, 3$).

(b) Draw the contours $\epsilon(\mathbf{k}) = \text{const.}$ in the (k_x, k_y)-plane for $d = 2$.

Hint: Diagonalize the Hamiltonian by a Fourier transform, $c_{j\sigma} = \frac{1}{\sqrt{N}} \sum_{\mathbf{k}} \exp(i\mathbf{k}\mathbf{r}_j) c_{\mathbf{k}\sigma}$, here, $\mathbf{r}_j$ are the coordinates of the lattice sites; N is their total number.

(7) **Wigner function** (5 Punkte)
Knowing the density operator $\hat{\rho}$ of a particle is equivalent to knowing its density matrix $\rho(\mathbf{x}, \mathbf{x}') = \langle \mathbf{x} | \hat{\rho} | \mathbf{x}' \rangle$ in the position representation (Hint: partition of unity).

We now consider a one-dimensional situation (the generalization is easy) and use $\rho(x, x')$ to define the *Wigner function*

$$f(r, p) = \frac{1}{2\pi\hbar} \int \mathrm{d}y \exp(ipy/\hbar) \rho(r + \frac{y}{2}, r - \frac{y}{2}) .$$

Show that $f(r, p)$ has the following properties:

(a) $f(r, p)$ is real.

(b) $\int \mathrm{d}p f(r, p)$ is the correct quantum-mechanical probability density in position space.

(c) $\int \mathrm{d}r f(r, p)$ is the correct quantum-mechanical probability density in momentum space.

(d) Hence $f(r, p)$ looks like a classical phase-space distribution that reproduces quantum mechanics, which appears to be a contradiction to everything we know (e.g., the uncertainty relation).
Calculate and plot $f(r, p)$ for the one-dimensional harmonic oscillator prepared in its n-th eigenstate for $n = 0, 1, 2$ to see what is the problem. Use a computer if necessary.