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Quantenmechanik, Herbstsemester 2025

Blatt 10 = letztes "offizielles" Übungsblatt
(d.h., 50% der Hausaufgaben = 50 Punkte)

Abgabe: 25.11.25, 12:00H (Treppenhaus 4. Stock)
Tutor: Niels Lörch, Zi.: 4.10

(1) **Two spins with time-dependent coupling** (4 Punkte)
Consider two coupled spin 1/2 particles with a time-dependent coupling constant $J(t)$ which approaches zero for $t \to \pm\infty$. The Hamiltonian is

$$H(t) = J(t)\mathbf{S}_1 \cdot \mathbf{S}_2 \,.$$

Assume that the system is prepared in the state $|\psi(t \to -\infty)\rangle = |\uparrow\downarrow\rangle_z := |\uparrow\rangle_z^{(1)} |\downarrow\rangle_z^{(2)}$.

(a) Does H commute with itself at different times?
Write down an expression for the time evolution operator $U(t_\mathrm{f}, -\infty)$ and express it in terms of $\alpha(t_\mathrm{f}) := \hbar \int_{-\infty}^{t_\mathrm{f}} \mathrm{d}t\, J(t)$ which is assumed to be finite.

(b) Obtain an explicit exact expression of the state $|\psi(t_\mathrm{f})\rangle$ at time $t = t_\mathrm{f}$.
Hint: $\mathbf{S}_1 \cdot \mathbf{S}_2 = \frac{1}{2}[(\mathbf{S}_1 + \mathbf{S}_2)^2 - \mathbf{S}_1^2 - \mathbf{S}_2^2]$.

(c) What is the probability to find the system in the state $|\downarrow\uparrow\rangle_z$ for $t = t_\mathrm{f}$?

(d) [**independent of (a) – (c)**]
Repeat (c) using first-order time-dependent perturbation theory, i.e., calculate the probability $P_{|\uparrow\downarrow\rangle \to |\downarrow\uparrow\rangle}(t = t_\mathrm{f})$.
If you have solved (c), compare with the exact result.

(2) **One-dimensional toy model for the photoelectric effect** (3 Punkte)
Consider an electron bound in an attractive δ-function potential, $H_0 = \frac{-\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \alpha\delta(x)$. Calculate the probability per unit time of "ionization" if the electron is under the influence of a harmonically varying electric field, i.e., a perturbation $V(x,t) = -xeE_0 \cos\omega t$.

(a) Solve the problem assuming that the final states do not "see" the δ-function potential (i.e., assume that the final states are plane waves).
Hint: Golden rule. The ground state of H_0 was found in problem 4 of Blatt 5: $\psi_0(x) = \sqrt{\kappa}e^{-\kappa|x|}$ where $\kappa = \frac{m\alpha}{\hbar^2}$; the ground-state energy is $\frac{-\hbar^2\kappa^2}{2m}$.

(b) Repeat (a) taking into account the influence of the δ-function potential on the final states.

(3) **Wigner-Eckart theorem** (3 Punkte)
Electromagnetic quadrupole transitions in the hydrogen atom are described by matrix elements of the (spherical) quadrupole operators $Q_m^{(2)} \sim r^2 Y_{2m}$ that form a set of spherical tensor operators.

(a) Calculate the ratio B/A of the following matrix elements; here, $|nlm\rangle$ are the eigenstates of the hydrogen atom:

$$A = \langle n'43|Q_2^{(2)}|n21\rangle \, ,$$
$$B = \langle n'4,-2|Q_0^{(2)}|n2,-2\rangle \, .$$

(b) Calculate

$$C = \langle n'51|Q_2^{(2)}|n1,-1\rangle \, ,$$
$$D = \langle n'31|Q_0^{(2)}|n1,-1\rangle \, .$$

(c) Consider the matrix element $\langle 4lm|z(x+iy)|n21\rangle$ where x,y,z are Cartesian coordinates. Which values for l and m are allowed, i.e., lead to non-vanishing values?

(4) **Coherent states** (4 Bonuspunkte)
Consider a harmonic oscillator with mass m and frequency ω, and let $|n\rangle$ denote the n-th eigenstate ($n = 0, 1, 2, \ldots$). Furthermore, $\hat{a}^\dagger$ and $\hat{a}$ denote the creation and annihilation operators of the harmonic oscillator, respectively.

Consider now a so-called "*coherent state*" parametrized by the complex number α,

$$|\alpha\rangle = |\alpha(t=0)\rangle = \; e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle = e^{-|\alpha|^2/2} e^{\alpha \hat{a}^\dagger} |0\rangle.$$

(a) Prove the last equality in the above equation.

(b) Show that $|\alpha\rangle$ is an eigenstate of the annihilation operator $\hat{a}$, and calculate the eigenvalue. Show further that $|\alpha\rangle$ is *not* an eigenstate of $\hat{a}^\dagger$. Can $\hat{a}^\dagger$ have eigenstates?

(c) Show that the probability to measure the energy $\hbar\omega(n + 1/2)$ is distributed according to the Poisson distribution $P_n = e^{-\mu} \frac{\mu^n}{n!}$, with an appropriately chosen parameter μ. Show further that $|\alpha\rangle$ is normalized and calculate the energy expectation value and variance.
Hint: The result in (b) may be useful.

(d) Calculate the expectation values $\langle \hat{x} \rangle$ and $\langle \hat{p} \rangle$ of the position and momentum in this state.

(e) Show that $|\alpha(t > 0)\rangle$ is also a coherent state (apart from a phase-factor) and calculate the complex amplitude $\alpha(t)$. Use this fact (and the result in (d)) to calculate the time evolution of the expectation values $\langle \hat{x} \rangle(t)$ and $\langle \hat{p} \rangle(t)$ and compare with the Ehrenfest theorem that was discussed in the lecture.