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Mechanik, Herbstsemester 2024

Blatt 8

Abgabe: 12.11.2024, 12:00H, **entweder auf adam in den entsprechenden Ordner, oder in das Fach im Treppenhaus 4. Stock!**
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(1) **Bottle on the floor of a tram** (2 Punkte)
A bottle (modeled as a homogeneous circular cylinder of radius R and mass M) lies on the floor of a tram; its orientation is perpendicular to the tram's direction of motion. When the tram starts moving with acceleration a, the bottle will start to roll (we assume without sliding).

(a) Calculate the moment of inertia I along the symmetry axis. Result: $I = \frac{1}{2}MR^2$

(b) Determine the acceleration $\tilde{a}$ of the bottle relative to the passengers.

(2) **Inertia ellipsoid** (2 Punkte)
Consider a body with density $\rho(\mathbf{x})$ and inertia tensor $I_{\mu\nu}$. We define the set Γ of all points $\boldsymbol{\omega}$ for which $\boldsymbol{\omega}^T I \boldsymbol{\omega} = 1$.

(a) Show that Γ is an ellipsoid whose axes are parallel to the principle axes of the body and whose (semi-)axes lengths are $1/\sqrt{I_i}$ where I_i, $i = 1, 2, 3$ are the principle moments of inertia of the body.

(b) Assume that the body is invariant under a symmetry transformation R, $\rho(R\mathbf{x}) = \rho(\mathbf{x})$. Show that Γ is also invariant under R. Hint: $R^{\mathrm{T}}R = 1$.

(c) Show: if the body has a k-fold symmetry axis (i.e., is invariant under rotations by $2\pi/k$ about this axis) with $k \geq 3$, this axis is a principle axis and the body is a symmetric top. If the body has more than one such axis, it is a spherical top.

(3) **Free symmetric top in the body frame and in the lab frame** (3 Punkte)
In the lecture we solved Euler's equation for a free (i.e., no torques) symmetric top ($I_1 = I_2 =: I \neq I_3$).

(a) Describe and sketch the motion of the angular momentum $\mathbf{L}$, the angular velocity $\boldsymbol{\omega}$, and the principal axis $\mathbf{e}_3$ as seen in the body frame. Distinguish the cases $I_3 > I$ and $I_3 < I$ (which case corresponds to a coin-like shape, which one to a carrot-like shape?).

(b) Now we want to analyze the motion in the lab frame. In terms of the (changing) principal axes $\mathbf{e}_i$, we have

$$\boldsymbol{\omega} = (\omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2) + \omega_3 \mathbf{e}_3$$
$$\mathbf{L} = I(\omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2) + I_3 \omega_3 \mathbf{e}_3 \, .$$

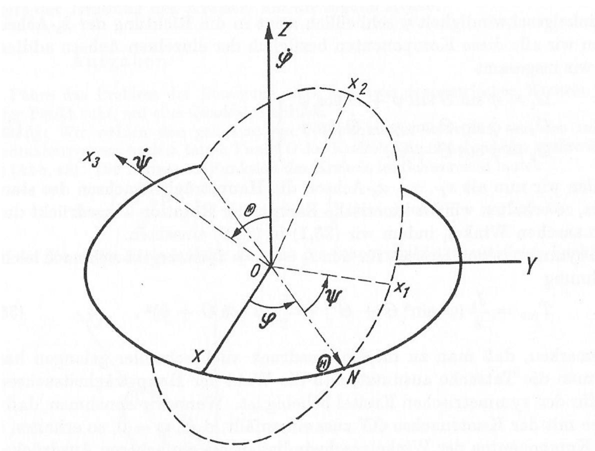
Eliminate $(\omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2)$ from these two equations and express $\boldsymbol{\omega}$ in terms of $\mathbf{L}$ and $\mathbf{e}_3$. Result: $\boldsymbol{\omega} = \frac{\mathbf{L}}{I} - \xi \mathbf{e}_3$ where $\xi = \omega_3 \frac{I_3 - I}{I}$.
Conclude that $\mathbf{L}$, ω, and $\mathbf{e}_3$ lie in a plane.
Since $\mathbf{e}_3$ is fixed in the body frame, its change in the lab frame comes only from rotation around $\boldsymbol{\omega}$, i.e., $\frac{d\mathbf{e}_3}{dt} = \boldsymbol{\omega} \times \mathbf{e}_3$. Conclude that $\mathbf{e}_3$ precesses around $\tilde{\boldsymbol{\omega}} = \mathbf{L}/I$ (that is constant in the lab frame - why?) with frequency $\tilde{\omega} = \frac{L}{I}$.
Describe and sketch the motion of $\mathbf{L}$, $\boldsymbol{\omega}$, and $\mathbf{e}_3$ as seen in the lab frame. Distinguish the cases $I_3 > I$ and $I_3 < I$.

(c) (Bonus points) We found that a person standing on the rotating body sees $\mathbf{L}$ (and $\boldsymbol{\omega}$) precess with frequency $\xi = \omega_3 \frac{I_3 - I}{I}$ around $\mathbf{e}_3$. A person in the lab frame sees $\mathbf{e}_3$ (and $\boldsymbol{\omega}$) precess with frequency $\frac{L}{I}$ around $\mathbf{L}$. Are these two facts compatible?

(4) **Angular velocity in the body system and Euler angles** (3 Punkte)
The angular velocity $\boldsymbol{\omega}$ in the body frame can be expressed in terms of the time-dependent Euler angles by projecting the Euler rotations onto the body axes. Show that

$$\boldsymbol{\omega} = \begin{pmatrix} \dot{\varphi} \sin\theta \sin\psi + \dot{\theta} \cos\psi \\ \dot{\varphi} \sin\theta \cos\psi - \dot{\theta} \sin\psi \\ \dot{\varphi} \cos\theta + \dot{\psi} \end{pmatrix} .$$

Hint: Use $\boldsymbol{\omega} = \dot{\varphi} \mathbf{e}_z + \dot{\theta} \mathbf{e}_{ON} + \dot{\psi} \mathbf{e}_3$ and express the unit vectors $\mathbf{e}_z$ and $\mathbf{e}_{ON}$ pointing in the direction of the z-axis and the axis ON in terms of the unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ pointing in the direction of the body-fixed axes.



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