Departement Physik, Universität Basel
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Mechanik, Herbstsemester 2024

Blatt 7

Abgabe: 5.11.2024, 12:00H, **entweder auf adam in den entsprechenden Ordner, oder in das Fach im Treppenhaus 4. Stock!**
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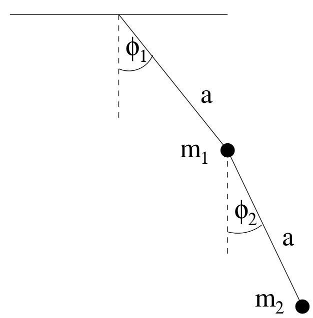
(1) **Double pendulum** (5 Punkte)
Consider a planar double pendulum (masses m_1, m_2, same length a) subject to a homogeneous gravitational field.



(a) Construct the Lagrangian by writing down the kinetic and potential energies of m_1 and m_2. Result: $L(\phi_1, \phi_2, \dot{\phi}_1, \dot{\phi}_2) = \frac{1}{2}(m_1 + m_2)a^2\dot{\phi}_1^2 + \frac{1}{2}m_2 a^2 \dot{\phi}_2^2$
$+ m_2 a^2 \cos(\phi_1 - \phi_2)\dot{\phi}_1\dot{\phi}_2 + (m_1 + m_2)ga\cos\phi_1 + m_2 ga\cos\phi_2$.

(b) Expand the Lagrangian for small oscillations $\phi_1, \phi_2 \ll 1$. Keep terms up to quadratic order in ϕ_i, $\dot{\phi}_i$. Result: $L_{\text{small}} = \frac{1}{2}\sum_{i,j}^{2} m_{ij}\dot{\phi}_i\dot{\phi}_j - \frac{1}{2}\sum_{i,j}^{2} k_{ij}\phi_i\phi_j$
where $M = m_{ij} = a^2 \begin{pmatrix} m_1 + m_2 & m_2 \\ m_2 & m_2 \end{pmatrix}$, $K = k_{ij} = ga \begin{pmatrix} m_1 + m_2 & 0 \\ 0 & m_2 \end{pmatrix}$.

(c) Write down the equations of motion.

(d) Calculate the eigenfrequencies and eigenmodes. Discuss and plot them as a function of m_2/m_1 and discuss their limiting behavior. What happens for

i. $m_2/m_1 \ll 1$? (don't simply set $m_2 = 0$)

ii. $m_2/m_1 \to \infty$?

(e) Calculate and discuss the time dependence of the motion for $m_1 = m_2$ and the initial condition $\phi_1 = 0$, $\phi_2 = \alpha \neq 0$, $\dot{\phi}_1 = \dot{\phi}_2 = 0$.

(2) **Inertia tensor** (3 Punkte)
Determine the inertia tensor of the following bodies and express them through the total mass M and the geometrical parameters. Assume that the origin of the coordinate system coincides with the body's center of mass.

(a) Thin rod of length l (the thickness of the rod can be neglected).

(b) Homogeneous sphere of radius R.

(c) How is the result of (b) modified if an additional pointlike mass δm is added somewhere on the surface of the sphere? Keep the origin of the coordinate system in the center of the sphere.

(3) **Steiner's theorem (aka Parallel axis theorem)** (2 Punkte)
Consider a rigid body of total mass M whose inertia tensor I_{ij}^S is given in the rigid-body coordinate system x_1, x_2, x_3 whose origin coincides with the center of mass S. If we shift this coordinate system by a vector $\mathbf{a}$, we obtain a new coordinate system x_1', x_2', x_3' whose axes are parallel to the axes of x_1, x_2, x_3.

Show that the inertia tensor in the new coordinate system has the form

$$I_{ij} = I_{ij}^S + M(\mathbf{a}^2 \delta_{ij} - a_i a_j) \, .$$

(4) **Bonus problem: resonance curve of an anharmonic oscillator** (5 Bonuspunkte)
The goal of this problem is to numerically study the resonance curve of an anharmonic oscillator. The (dimensionless) equation of motion of the oscillator is assumed to be

$$\frac{d^2x}{dt^2} = -x(1+x^2) - \gamma \frac{dx}{dt} + A \sin(\varphi(t)) \, . \tag{1}$$

Here, γ is the friction constant and A the amplitude of the driving force. To obtain the resonance curve, we vary the phase $\varphi(t)$ during the time T such that the "instantaneous" frequency $\dot{\varphi}$ of the driving force grows linearly with time from 0 to $\omega_{\max}$:

$$\varphi(t) = \frac{\omega_{\max}}{2T} t^2$$

for $t \in [0, T]$.

(a) Give the analytical expression for $x(t)$ for a harmonic oscillator (i.e., in the absence of the term $\sim x^3$) subject to a drive $A \sin(\omega t)$ with a fixed frequency ω.

(b) Integrate the equation of motion (1) numerically, e.g., using the parameters $\gamma = 0.5$, $T = 3000$ (or even larger), $\omega_{\max} = 2$, and three different drive amplitudes $A = 0.1$, $A = 0.5$, and $A = 1.5$. The initial condition is not important (why?). Plot $x(t)$ and – as a comparison – the amplitude $x_{\max}(\omega(t))$ of the harmonic case found in (a).

(c) Determine the resonance curve for the largest drive amplitude, $A = 1.5$, by going "backwards" from $\omega_{\max}$ to 0 (how do you have to choose $\varphi(t)$ in this case?). Determine the frequency interval in which there is *hysteresis*, i.e., two different amplitudes of the oscillation at a given value of the drive amplitude A in the forward and backward directions.

(d) Give a qualitative explanation for the fact that at large A, you can observe a second peak in the resonance curve at smaller frequencies.