Departement Physik, Universität Basel
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Mechanik, Herbstsemester 2024

Blatt 5

Abgabe: 22.10.2024, 12:00H, **entweder auf adam in den entsprechenden Ordner, oder in das Fach im Treppenhaus 4. Stock!**
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(1) **Lenz-Runge vector** (3 Punkte)

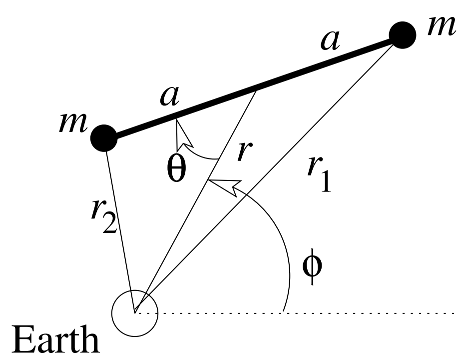
(a) Show that for the potential $V(r) = -\alpha/r$ there is an additional constant of motion (apart from the energy E and the angular momentum $\mathbf{L}$), the Lenz-Runge vector

$$\mathbf{A} = \dot{\mathbf{x}} \times \mathbf{L} - \alpha \frac{\mathbf{x}}{|\mathbf{x}|} .$$

(b) What is the direction of $\mathbf{A}$? What happens if the orbit is circular?

(c) Calculate $\mathbf{x} \cdot \mathbf{A}$ and use this relation to determine the orbit. Compare with the result shown in the lecture. What is the geometrical significance of $|\mathbf{A}|/\alpha$?

(2) **Motion of a satellite** (4 Punkte)
We would like to model the motion of the ISS (International Space Station) in the Earth's gravitational field $V(r) = -m\tilde{\alpha}/r$. Consider a toy model of a dumbbell consisting of two point masses $m_1 = m_2 = m$ and a rigid massless connection (length $2a$). We assume that the motion is planar; the center-of-mass motion can be described by polar coordinates r, ϕ, and the orientation of the dumbbell by the angle θ, see figure.



(a) Write down the Lagrangian L and use it to derive the equations of motion.
Result for L: $L = m(\dot{r}^2 + r^2\dot{\phi}^2 + a^2(\dot{\phi} - \dot{\theta})^2) + m\tilde{\alpha}\left(\dfrac{1}{r_2(r,\theta)} + \dfrac{1}{r_1(r,\theta)}\right)$.

(b) Show that there are solutions of the form $r = r_0 =$ const. and $\dot{\phi} =$ const., and either $\theta = 0$ or $\theta = \pi/2$. Discuss these two solutions. Give expressions for their periods and expand them in the small parameter a/r_0. Compare with Kepler's result.

(c) Bonus points: Integrate the equations of motions numerically for initial conditions $\theta(t=0) \notin \{0, \pi/2\}$ and discuss your results.

(3) **Scattering from a Lennard-Jones potential** (3 Punkte + Bonuspunkte)
The Lennard-Jones potential is defined by

$$V(r) = 4\epsilon\left(\left(\frac{\beta}{r}\right)^{12} - \left(\frac{\beta}{r}\right)^{6}\right).$$

(a) Plot the effective potential $V_{\text{eff}}(r)$ for different values of the impact parameter s (i.e., the perpendicular distance between the center of force and the incident velocity). Discuss the qualitatively different cases. What are the characteristic energies?

(b) Find and plot typical trajectories $(r(t), \varphi(t))$:
Find $r(t)$ by numerically solving $\ddot{r} = -\frac{\partial V_{\text{eff}}(r)}{\partial r}$. Take $\epsilon = \beta = 2E = m = 1$.
Find $\varphi(t)$ by using the conservation of angular momentum $\ell = mr^2\dot{\varphi}$.

(c) Sketch the scattering angle $\Theta(s, E)$ as a function of the impact parameter for the energies found in (a). Give reasons for the shape of the curves and discuss them.

(4) **Bonus problem: Brachistochrone** (3 Bonuspunkte)
We return to the situation of problem 1 of Blatt 1: a point mass that slides without friction on a curve $y(x)$ in the xy-plane connecting the two points $r_A = (0,0)$ and $r_B = (2,-1)$. The mass starts at r_A with velocity 0 and is subject to the Earth's gravitational field that is assumed to be homogeneous and point in the negative y-direction. The total time that the particle needs to reach r_B can be expressed as $T = \int_{r_{Ax}}^{r_{Bx}} \mathrm{d}x F(y, y')$ where $F(y, y') = \frac{\sqrt{1+y'^2}}{\sqrt{2g(-y)}}$. We want to find the curve that minimizes T (Brachistochrone = ancient Greek for "shortest time").

(a) Use Lagrange's equations to write down a differential equation for the solution.

(b) Solve the differential equation and adapt the constants such that the boundary conditions are fulfilled.

(c) Compare with the results of problem 1 of Blatt 1.

Remark: This problem was first posed and solved in Basel in 1696 by Johann Bernoulli!