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Mechanik, Herbstsemester 2024

Blatt 10

Abgabe: 26.11.2024, 12:00H, **entweder auf adam in den entsprechenden Ordner, oder in das Fach im Treppenhaus 4. Stock!**
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(1) **Canonical transformations** (5 Punkte)

We consider a system with one degree of freedom ($f = 1$) and want to study canonical transformations from (q, p) to (Q, P) generated by $F_2(q, P, t)$ which leads to

$$Q = \frac{\partial F_2}{\partial P} \qquad\qquad p = \frac{\partial F_2}{\partial q} \,.$$

Construct suitable generators F_2 for the following cases:

(a) Identical transformation: $Q = q$, $P = p$.

(b) Galilei transformation, for which Q, P belong to a system moving with velocity v with respect to the original system: $q = Q + vt$, $p = P + mv$. Write down the new Hamiltonian $K(Q, P)$ that is related to the original Hamiltonian H by the relation $K = H + \frac{\partial F_2}{\partial t}$.

(c) A general point transformation from q to the new coordinate Q, i.e., $Q = g(q, t)$. Construct a generator F_2 such that the new momentum P is *proportional* to the original momentum p. Show that the transformation rule for the momentum can also be obtained from the fact that the Lagrangian is invariant under a point transformation: $L(q, \dot{q}; t) = \tilde{L}(Q, \dot{Q}; t)$.

(2) **Phase portraits and Liouville's theorem** (5 Punkte)

Liouville's theorem states that the volume in phase space occupied by a collection of systems remains constant over time.

Consider the dimensionless Hamiltonian $H(q, p) = \dfrac{p^2}{2} + V(q)$, where $V(q)$ is given by

(a) $V(q) = \dfrac{q^2}{2}$ (harmonic oscillator).

(b) $V(q) = \dfrac{q^2}{2} + q^4$ (anharmonic oscillator).

(c) $V(q) = 1 - \cos(q)$ (pendulum; q corresponds to the deflection angle).

For each of these examples, draw phase portraits by plotting phase-space orbits $(q(t), p(t))$ for a number of equidistant energies. Discuss the difference between open and closed orbits in (c). What happens for the initial condition $q = 0$, $p = 2$ in (c)?

Consider now the rectangular phase-space volume $-\frac{1}{2} \leq q \leq \frac{1}{2}, 1 \leq p \leq 3)$ at $t = 0$. Analyze its time evolution qualitatively by carefully sketching (or numerically calculating) its shape for different times (at least for a time $t \approx 1$ and for a time $t \gg 1$).