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Quantenmechanik, Herbstsemester 2023

Blatt 7

Abgabe: 7.11.23, 12:00H (Treppenhaus 4. Stock)
Tutor: Tobias Nadolny, Zi. 4.48

(1) **Properties of spherical harmonics** $Y_{lm}(\theta, \phi)$ (2 Punkte + 1 Bonuspunkt)

(a) Calculate $\int d\Omega \, Y_{31}^*(\theta, \phi) \sin(\theta) \sin(\phi)$.

(b) A particle is described by the wave function $\psi(\mathbf{r}) = C(2x + iy + z)e^{-ar}$ where C is a normalization constant. We measure L_z. What is the probability to find $m = -1, 0, 1$?

(c) Bonus point: Expand $f(\theta, \phi) = \sin(\theta)$ in terms of spherical harmonics.

(2) **Infinite 3D spherical potential well** (3 Punkte)
Consider a particle of mass m in an infinite three-dimensional potential well, described by the spherically-symmetric potential

$$V(r) = \begin{cases} 0, & r < a \, ; \\ \infty \, , & r \geq a \, . \end{cases}$$

Determine the energy spectrum of this system.
In particular, determine the approximate values and the ordering of the six lowest energy levels of the system in terms of the principal quantum number n ($n = 1, 2, 3, \ldots$) and the orbital quantum number l ($l = 0, 1, 2, 3, \ldots$).

(3) **Molecular magnet** (3 Punkte + 2 Bonuspunkte)
We consider a (hypothetical) molecular magnet with spin S that is described by the Hamiltonian

$$H = -AS_z^2 + B(S_x^2 - S_y^2) \, ,$$

(a) We first assume $S = 2$. Construct the spin matrices S_z, S_+, S_-, S_x, and S_y.

(b) For $S = 2$, calculate the energy eigenvalues and eigenstates for $A > 0$ and arbitrary B, either analytically or by using a computer. Discuss your results.
Hint: Show that the Hilbert space decays into two subspaces that are invariant under H.

(c) Bonus points: Repeat your calculation for general S and compare the cases of integer and half-integer S.

(4) **Penning trap** (2 Punkte + 2 Bonuspunkte)

In a Penning trap a charged particle moves in a superposition of a uniform magnetic field $\mathbf{B}$ directed along the z-axis and a quadrupole electric field that leads to a contribution $V(\mathbf{r}) = C(2z^2 - x^2 - y^2)$ to the Hamiltonian; $C =: \frac{1}{4}m\omega_0^2$ is a positive constant. We consider an electron of charge $-e$ ($e > 0$) with g-factor $g = 2(1 + a)$; the Hamiltonian is then given by

$$H = \frac{1}{2m}(\mathbf{p} + e\mathbf{A}(\mathbf{r}))^2 + V(\mathbf{r}) + (1 + a)\frac{e}{m}\mathbf{S} \cdot \mathbf{B}\,. \tag{1}$$

It is convenient to choose the gauge $\mathbf{A}(\mathbf{r}) = \frac{1}{2}\mathbf{B} \times \mathbf{r}$.

(a) Calculate and sketch the electric field.

(b) Show that the Hamiltonian can be split into three terms: $H = H_z + H_t + H_s$, where

$$\begin{aligned} H_z &= \frac{p_z^2}{2m} + \frac{1}{2}m\omega_0^2 z^2 \\ H_t &= \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2}m\Omega^2(x^2 + y^2) + \frac{1}{2}\omega_c L_z \\ H_s &= (1 + a)\omega_c S_z\,. \end{aligned}$$

Here, $\omega_c = eB/m$ is the cyclotron frequency that we assume to be much larger than ω_0. Express Ω as a function of ω_c and ω_0.

(c) Describe the motion in the z-direction.

(d) Now we would like to understand the transverse motion in the xy-plane. Define the right and left circular annihilation operators by

$$\begin{aligned} a_r &= \frac{1}{2}\left(\beta(x - iy) + \frac{i}{\beta\hbar}(p_x - ip_y)\right) \\ a_l &= \frac{1}{2}\left(\beta(x + iy) + \frac{i}{\beta\hbar}(p_x + ip_y)\right) \end{aligned}$$

where β is a real constant. Show that $[a_r, a_r^\dagger] = 1$, $[a_l, a_l^\dagger] = 1$, and that "mixed" commutators vanish.

(e) Show that $L_z = \hbar(N_r - N_l)$ where $N_r := a_r^\dagger a_r$ and $N_l := a_l^\dagger a_l$.

(f) Choose β such that $H_t = \hbar\omega_c'(N_r + \frac{1}{2}) - \hbar\omega_m(N_l + \frac{1}{2})$. Calculate ω_c' and ω_m in terms of ω_c and ω_0.

(g) Write down the complete energy spectrum of the Hamiltonian H.