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Quantenmechanik, Herbstsemester 2023

Blatt 6

Abgabe: 31.10.23, 12:00H (Treppenhaus 4. Stock)
Tutor: Tobias Nadolny, Zi. 4.48

(1) **Components of the angular momentum operator** (3 Punkte)
Using the commutation relation $[x_j, p_k] = i\hbar\delta_{jk}$, show the following relations for the orbital angular momentum operator $\mathbf{L} = (L_x, L_y, L_z)$; $\mathbf{n}$ is a real vector.

(a) $[\mathbf{n} \cdot \mathbf{L}, \mathbf{r}] = i\hbar \mathbf{r} \times \mathbf{n}$

(b) $[\mathbf{n} \cdot \mathbf{L}, \mathbf{p}] = i\hbar \mathbf{p} \times \mathbf{n}$

(c) $\mathbf{L} \times \mathbf{L} = i\hbar \mathbf{L}$

(2) **Can l be half-integer for orbital angular momenta?** (3 Punkte)
In this problem, we will prove that the eigenvalues m of the angular momentum operator $L_z/\hbar$ must be integer. Therefore, the orbital angular momentum quantum number l can take only integer values.

(a) Express the operator L_z in terms of the creation and destruction operators, $a_i^\dagger$ and a_i $(i = 1, 2, 3)$ by using the transformations

$$x_i = \sqrt{\frac{\hbar}{2m\omega}}(a_i + a_i^\dagger) \, ; \qquad p_i = -i\sqrt{\frac{\hbar m\omega}{2}}(a_i - a_i^\dagger) \, .$$

(b) By introducing new operators b_1, b_2 (and their hermitian conjugates) that are linear combinations of the a_i's, show that L_z can be written in the form

$$L_z = \hbar(b_2^\dagger b_2 - b_1^\dagger b_1) \, ,$$

where the operators b_1, b_2 satisfy the commutation relations $[b_1, b_1^\dagger] = [b_2, b_2^\dagger] = 1$, $[b_i, b_i] = [b_i^\dagger, b_i^\dagger] = 0$ for $i = 1, 2$.

(c) Argue that the eigenvalues of L_z should be an integer multiplied by $\hbar$ and consequently that the orbital angular momentum l should be integer.

(3) **Kronig-Penney-Model for $V_0 < 0$** (4 Punkte + 2 Bonuspunkte)
In the lecture we discussed the Kronig-Penney-Model

$$V(x) = V_0 \sum_{n=-\infty}^{\infty} \delta(x - na) \, .$$

Solving the equation

$$\cos(ka) = \cos(qa) + \frac{mV_0 a}{\hbar^2} \frac{\sin(qa)}{qa} \qquad (1)$$

graphically for $V_0 > 0$, we obtained the allowed and forbidden values of q which resulted in energy bands and energy gaps in $\epsilon_k = \hbar^2 q^2/(2m)$.

Consider now the case $V_0 < 0$.

(a) Sketch the right-hand side of Eq. (1) as a function of qa and solve the equation graphically or with a computer. Sketch the lowest energy band ϵ_k.

(b) In the case $V_0 < 0$ there are solutions to the Schrödinger equation with *negative* energy eigenvalues. What does this imply for q? Solve Eq. (1) for this case graphically or with a computer. Sketch the part of the lowest energy band that originates from this solution and complete the sketch in (a). Interpret your result.

(c) Bonus points: Calculate and plot the first four energy bands numerically for $mV_0a/\hbar^2 = -1, -2, -5$.

(4) **Supersymmetry** (5 Bonuspunkte)

Consider the two operators

$$A = i\frac{p}{\sqrt{2m}} + W(x) \quad \text{and} \quad A^\dagger = -i\frac{p}{\sqrt{2m}} + W(x)$$

for some function $W(x)$; here, p is the momentum operator. Using these two operators we can construct two Hamiltonians,

$$H_1 = A^\dagger A = \frac{p^2}{2m} + V_1(x) \quad \text{and} \quad H_2 = AA^\dagger = \frac{p^2}{2m} + V_2(x)\,.$$

$W(x)$ is called *superpotential*; V_1 and V_2 are called *supersymmetric partner potentials*.

(a) Find the potentials $V_1(x)$ and $V_2(x)$ in terms of $W(x)$.
Hint: Apply $A^\dagger A$ to a wave function $\psi(x)$.

(b) Show that if $|\psi_n^{(1)}\rangle$ is an eigenstate of H_1 with eigenvalue $E_n^{(1)}$, then $A|\psi_n^{(1)}\rangle$ is an eigenstate of H_2 with the same eigenvalue. Similarly, show that if $|\psi_n^{(2)}\rangle$ is an eigenstate of H_2 with eigenvalue $E_n^{(2)}$, then $A^\dagger|\psi_n^{(2)}\rangle$ is an eigenstate of H_1 with the same eigenvalue. The two Hamiltonians therefore have essentially identical spectra.

(c) One ordinarily chooses $W(x)$ such that the ground state of H_1 satisfies $A|\psi_0^{(1)}\rangle = 0$ and hence $E_0^{(1)} = 0$. Use this to find $W(x)$ in terms of the ground state wave function $\psi_0^{(1)}(x)$. (The fact that A annihilates $|\psi_0^{(1)}\rangle$ means that H_2 has one less eigenstate that H_1 and is missing the eigenvalue $E_0^{(1)}$.)

(d) Consider the attractive δ-function potential $V_1(x) = \frac{m\alpha^2}{2\hbar^2} - \alpha\delta(x)$ with $\alpha > 0$. (The constant shift guarantees that $E_0^{(1)} = 0$.) As we saw in problem 2 of Blatt 5, it has a single bound state,

$$\psi_0^{(1)}(x) = \frac{\sqrt{m\alpha}}{\hbar}\exp\left(-\frac{m\alpha}{\hbar^2}|x|\right).$$

Determine $W(x)$ and the partner potential $V_2(x)$ and compare the properties of V_1 and V_2.

(e) Bonus points: Find and discuss the supersymmetric partner of the box with hard walls, $V_1(x) = -\frac{\pi^2\hbar^2}{2mL^2}$ for $|x| \leq \frac{L}{2}$ and ∞ otherwise.