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Quantenmechanik, Herbstsemester 2023

Blatt 5

Abgabe: **Tuesday 24.10.23**, 12:00H (Treppenhaus 4. Stock)
Tutor: Parvinder Solanki, Zi. 4.48

(1) **More on the charged particle in a magnetic field** (4 Punkte + 1 Bonuspunkt)
We consider a particle (charge q) in a magnetic field $\mathbf{B}(\mathbf{r})$.

(a) Show that the (Heisenberg) equation of motion for the velocity $\mathbf{v}$ reads

$$\frac{d}{dt}\mathbf{v} = \frac{q}{m}(\mathbf{v} \times \mathbf{B}) + i\frac{q\hbar}{2m^2}\nabla \times \mathbf{B}\,. \tag{1}$$

Hint: Write the Hamiltonian in terms of velocities and use the commutation relation for the components of the velocity, $[v_k, v_l] = i\hbar\frac{q}{m^2}\epsilon_{kln}B_n$.
Interpret the terms on the right-hand side of Eq. (1).

(b) We now assume that $\mathbf{B} = (0, 0, B) = const$. Solve Eq. (1) and show that the particle performs a cyclotron motion around a center (x_0, y_0) as in the classical case.

(c) Show that the *guiding center coordinates* x_0, y_0 are constants of motion but do not commute. Interpretation?

(d) Show that the square of the radius of the cyclotron motion has a sharp value in a (Landau) energy eigenstate and conclude that the radius of the stationary states grows like $\sqrt{n}$ for large Landau level number n.

(e) Bonus point: How is Eq. (1) modified if there is an additional electrical potential term $+q\phi(\mathbf{x}, t)$ in the Hamiltonian?
Hint: To ensure that the new expression is gauge-invariant you have to allow an explicit time dependence of $\mathbf{A}$.

(2) **Gauge transformation in quantum mechanics** (2 Punkte)
Electric and magnetic fields do not appear in the Schrödinger equation of a charged particle with charge q. Instead, the vector potential $\mathbf{A}$ and the scalar potential Φ enter, where $\mathbf{E} = -\nabla\Phi - \partial_t\mathbf{A}$ and $\mathbf{B} = \nabla \times \mathbf{A}$. (Remember that both the fields and the potentials depend on $\mathbf{x}$ and t.) The potentials are not defined uniquely: a gauge transformation to new potentials $\mathbf{A}'$ und Φ' with

$$\mathbf{A}' \;=\; \mathbf{A} + \nabla\chi\,; \qquad \Phi' = \Phi - \partial_t\chi\,,$$

where $\chi(\mathbf{x}, t)$ is an arbitrary smooth function, will not change the fields.

(a) Prove the following operator equation (you should think of both sides of the equation as operators that act on a wave function standing on their right):

$$e^{f(y)}\frac{\partial}{\partial y} = \left(\frac{\partial}{\partial y} - \frac{\partial f}{\partial y}\right)e^{f(y)}\,.$$

(b) Assume that $\Psi(\mathbf{x},t)$ obeys the Schrödinger equation,

$$i\hbar\partial_t\Psi(\mathbf{x},t) = \frac{1}{2m}\left(\hat{\mathbf{p}} - q\mathbf{A}\right)^2\Psi(\mathbf{x},t) + q\Phi\Psi(\mathbf{x},t)\,.$$

Use (a) to show that the gauge-transformed wave function defined by

$$\Psi'(\mathbf{x},t) := \exp(iq\chi/\hbar)\Psi(\mathbf{x},t)$$

solves the modified Schrödinger equation that contains the transformed potential $\mathbf{A}'$ and Φ'.

(3) **Translation operator** (1 Punkt)
Show that the operator $\mathcal{T}_a \equiv \exp(ia\hat{p}/\hbar)$ induces a shift of the wave function in one-dimensional coordinate space, i.e.,

$$\langle x|\mathcal{T}_a|\psi\rangle = \langle x+a|\psi\rangle = \psi(x+a).$$

(4) **Binding $\boldsymbol{\delta}$-potential** (3 Punkte + 2 Bonuspunkte)
Consider a particle of mass m in the one-dimensional potential $V(x) = V_0\,\delta(x)$, with $V_0 < 0$.

(a) Find the bound state(s) (energy $E < 0$) of the particle in this potential. Sketch and interpret the wavefunction(s).
Hint: Either use the matching condition for wavefunctions in a δ-potential derived in the lecture, or go to momentum space.

(b) We now consider positive energies ($E > 0$). Make an ansatz for (unnormalized) scattering states. Derive the transmission amplitude $S(E)$ and the transmission probability $T(E)$ for a particle with energy E incident from the left.

(c) Bonus points: Re-derive the continuity equation for probability in the case of an imaginary potential. How do you interpret the role of imaginary potential in quantum mechanics? Next, consider an imaginary δ-potential $V(x) = \pm iV_0\,\delta(x)$ with $V_0 > 0$ real. Derive an expression for the absorption coefficient $\alpha(E) = 1 - T(E) - R(E)$, where $T(E)$ and $R(E)$ are the transmission and reflection probabilities for a particle with energy E incident from the left.
Remark: Non-hermitian Hamiltonians are a fashionable (and controversial) topic of current research.