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Quantenmechanik, Herbstsemester 2023

Blatt 10 = letztes "offizielles" Übungsblatt
(d.h., 50% der Hausaufgaben = 50 Punkte)

Abgabe: 28.11.23, 12:00H (Treppenhaus 4. Stock)
Tutor: Manel Bosch, Zi.: 2.12

(1) **Two spins with time-dependent coupling** (4 Punkte)
Consider two coupled spin 1/2 particles with a time-dependent coupling constant $J(t)$ which approaches zero for $t \to \pm\infty$. The Hamiltonian is

$$H(t) = J(t)\mathbf{S}_1 \cdot \mathbf{S}_2 \,.$$

Assume that the system is prepared in the state $|\psi(t \to -\infty)\rangle = |\uparrow\downarrow\rangle_z := |\uparrow\rangle_z^{(1)} |\downarrow\rangle_z^{(2)}$.

(a) Does H commute with itself at different times?
Write down an expression for the time evolution operator $U(t_\mathrm{f}, -\infty)$ and express it in terms of $\alpha(t_\mathrm{f}) := \hbar \int_{-\infty}^{t_\mathrm{f}} \mathrm{d}t \, J(t)$ which is assumed to be finite.

(b) Obtain an explicit exact expression of the state $|\psi(t_\mathrm{f})\rangle$ at time $t = t_\mathrm{f}$.
Hint: $\mathbf{S}_1 \cdot \mathbf{S}_2 = \frac{1}{2}[(\mathbf{S}_1 + \mathbf{S}_2)^2 - \mathbf{S}_1^2 - \mathbf{S}_2^2]$.

(c) What is the probability to find the system in the state $|\downarrow\uparrow\rangle_z$ for $t = t_\mathrm{f}$?

(d) [**independent of (a) – (c)**]
Repeat (c) using first-order time-dependent perturbation theory, i.e., calculate the probability $P_{|\uparrow\downarrow\rangle \to |\downarrow\uparrow\rangle}(t = t_\mathrm{f})$.
If you have solved (c), compare with the exact result.

(2) **One-dimensional toy model for the photoelectric effect** (3 Punkte)
Consider an electron bound in an attractive δ-function potential, $H_0 = \frac{-\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \alpha\delta(x)$. Calculate the probability per unit time of "ionization" if the electron is under the influence of a harmonically varying electric field, i.e., a perturbation $V(x,t) = -xeE_0 \cos\omega t$.

(a) Solve the problem assuming that the final states do not "see" the δ-function potential (i.e., assume that the final states are plane waves).
Hint: Golden rule. The ground state of H_0 was found in problem 4 of Blatt 5: $\psi_0(x) = \sqrt{\kappa}e^{-\kappa|x|}$ where $\kappa = \frac{m\alpha}{\hbar^2}$; the ground-state energy is $\frac{-\hbar^2\kappa^2}{2m}$.

(b) Repeat (a) taking into account the influence of the δ-function potential on the final states.

(3) **Wigner-Eckart theorem** (3 Punkte)

Electromagnetic quadrupole transitions in the hydrogen atom are described by matrix elements of the (spherical) quadrupole operators $Q_m^{(2)} \sim r^2 Y_{2m}$ that form a set of spherical tensor operators.

(a) Calculate the ratio B/A of the following matrix elements; here, $|nlm\rangle$ are the eigenstates of the hydrogen atom:

$$A = \langle n'43|Q_2^{(2)}|n21\rangle \, ,$$
$$B = \langle n'4,-2|Q_0^{(2)}|n2,-2\rangle \, .$$

(b) Calculate

$$C = \langle n'51|Q_2^{(2)}|n1,-1\rangle \, ,$$
$$D = \langle n'31|Q_0^{(2)}|n1,-1\rangle \, .$$

(c) Consider the matrix element $\langle 4lm|z(x+iy)|n21\rangle$ where x,y,z are Cartesian coordinates. Which values for l and m are allowed, i.e., lead to non-vanishing values?

(4) **Wigner function** (4 Bonuspunkte)

Knowing the density operator $\hat{\rho}$ of a particle is equivalent to knowing its density matrix $\rho(\mathbf{x}, \mathbf{x}') = \langle \mathbf{x}|\hat{\rho}|\mathbf{x}'\rangle$ in the position representation (Hint: partition of unity).

We now consider a one-dimensional situation (the generalization is easy) and use $\rho(x, x')$ to define the *Wigner function*

$$f(r,p) = \frac{1}{2\pi\hbar} \int \mathrm{d}y \exp(ipy/\hbar) \rho(r + \frac{y}{2}, r - \frac{y}{2}) \, .$$

Show that $f(r,p)$ has the following properties:

(a) $f(r,p)$ is real.

(b) $\int \mathrm{d}p f(r,p)$ is the correct quantum-mechanical probability density in position space.

(c) $\int \mathrm{d}r f(r,p)$ is the correct quantum-mechanical probability density in momentum space.

(d) Hence $f(r,p)$ looks like a classical phase-space distribution that reproduces quantum mechanics, which appears to be a contradiction to everything we know (e.g., the uncertainty relation).
Calculate and plot $f(r,p)$ for the one-dimensional harmonic oscillator prepared in its n-th eigenstate for $n = 0, 1, 2$ to see what is the problem. Use a computer if necessary.