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Mechanik, Herbstsemester 2022

Blatt 4

Abgabe: 18.10.2022, 12:00H, **entweder auf adam in den entsprechenden Ordner, oder in das Fach im Treppenhaus 4. Stock!**
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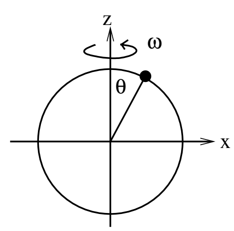
(1) **Circular cone revisited** (2 Punkte)
We would like to look at the particle moving on a circular cone (opening angle $2\alpha = \pi/2$) treated in problem 2 of Blatt 2 one more time: as we saw earlier, the Lagrangian in terms of the generalized (polar) coordinates r, φ is $L(r, \varphi, \dot{r}, \dot{\varphi}; t) = \frac{m}{2}(2\dot{r}^2 + r^2\dot{\varphi}^2) - mgr$. The variable φ is cyclic, hence $l_z = \partial L/\partial\dot{\varphi}$ is conserved. Also, since L does not depend explicitly on time, the energy $E = m\dot{r}^2 + l_z^2/(2mr^2) + mgr$ is conserved.

(a) Use E to write down a first-order differential equation for r and find the formal solution by separating the variables (you are not required to evaluate the integral).

(b) Discuss and sketch the allowed and forbidden regions by using the effective potential V_{eff}. Find the turning points of the radial coordinate where the energy is equal to V_{eff} and discuss your solution graphically. When does the equation have 0, 1, or 2 (physically relevant) solutions? Interpret the three different cases.

(2) **Bead on a rotating ring** (4 Punkte)
A bead (mass m) is subject to a (homogeneous) gravitational force acting in the negative z-direction and moves without friction on a vertical circle (radius R) that rotates with angular velocity ω about the z-axis, see figure. It is convenient to use the angle θ as the generalized coordinate (why do we need only one?).



(a) Write down the kinetic and potential energies and express the Lagrangian $L = T - V$ in terms of the generalized coordinate.
Result: $L(\theta, \dot{\theta}; t) = \dfrac{m}{2}R^2\dot{\theta}^2 + \dfrac{m}{2}R^2\omega^2\sin^2\theta - mgR\cos\theta$.

(b) Conclude that the motion of the bead corresponds to a one-dimensional motion in an effective potential $U(\theta)$. Sketch and discuss U for the two cases $R\omega^2/g < 1$ and $R\omega^2/g > 1$. What are the allowed regions for a given total energy? Determine and discuss the stable equilibrium position(s) of the bead in both cases.

(c) Carefully sketch the phase portraits (i.e., $\dot{\theta}$ as a function of θ) in both cases (or plot them using a computer).

(3) **Numerical experiments** (4 Punkte + bonus points)

The goal of this problem is to study the motion of a particle in a variety of two-dimensional potentials. Start by plotting the potential $V(x, y)$ (either as a 3D- or contour plot). Calculate the force on the particle and write down the two Newton equations (for a particle of mass m=1).

Solve the differential equations numerically, preferably using Julia (a "skeleton" is provided in the notebook folder on adam), or else your favorite method.

Use the following initial conditions: $\dot{y}(0) = 0.5$, 1, and 2, as well as always $x(0) = 1$, $y(0) = 0$, and $\dot{x}(0) = 0$. Plot the resulting trajectories ($x(t)$,$y(t)$) in the xy-plane in the time interval $[t = 0, t_{\text{end}}]$ for the values of t_{end} given below.

(a) $V(x, y) = -1/r$ where $r = \sqrt{x^2 + y^2}$ (Kepler problem of a particle in the gravitational field). What are the qualitatively different trajectories? ($t_{\text{end}} = 8$)

(b) $V(x, y) = \ln(r)$ (another central potential. It corresponds to the Coulomb potential of a charged wire perpendicular to the xy-plane). Discuss the qualitative differences to (a). Why can you find a circular orbit in both cases? ($t_{\text{end}} = 20$)

(c) $V(x, y) = x^2/2 + y^2$ (anisotropic two-dimensional harmonic oscillator). Why do the trajectories (so-called "Lissajous curves") not close? ($t_{\text{end}} = 20$)

(d) $V(x, y) = -(1 + \exp(10(\sin(x)^2 \sin(y)^2 - 1/2)))^{-1}$ (Chaotic motion in a quadratic lattice of scattering centers). Use the following two sets of initial conditions to test the influence of small changes of the initial conditions: $x(0) = 2$ or $x(0) = 2.1$, and always $y(0) = 0$, $\dot{x}(0) = 0$, $\dot{y}(0) = 0.5$. ($t_{\text{end}} = 40$)

(e) Bonus points: find another interesting potential and discuss the resulting motion in a qualitative way!

(f) Bonus points: consider a perturbed Kepler potential, $V(x, y) = -1/r + \beta/r^2$, where $\beta \ll 1$. Plot the trajectory for $\dot{y}(0) = 0.5$ and study the precession of the orbit as a function of β. The additional term looks very much like the centrifugal barrier term in the effective potential $V_{\text{eff}}(r)$. Why is it then that the additional force term causes a precession of the orbit, while an addition to the barrier, through a change in ℓ, does not?