# Mechanik, Herbstsemester 2022

### Blatt 2

**Schriftlicher Test: Montag, 19. Dezember 2022, 13.15 - 15 Uhr**
Hilfsmittel: Ein **hand**beschriebenes A4 Blatt.

**Mündliche Vorlesungprüfung (Examen): Mo. 30. Januar 2023**

(1) **Particle in a circular paraboloid** (3 Punkte)
A particle with mass $m$ slides on the inner surface of a circular paraboloid described by the equation $z = a(x^2 + y^2)$ and is subject to a (homogeneous) gravitational force acting in the negative z-direction.

(a) Describe the problem in cylindrical coordinates $(r, z, \varphi)$. Write down the equations of motion (they contain the constraint forces) for the two independent coordinates $r$ and $\varphi$.

(b) Eliminate the constraint forces and find the equations of motion for $r$ and $\varphi$. Result:

$$2\dot{r}\dot{\varphi} + r\ddot{\varphi} = 0 \tag{1}$$
$$\ddot{r}(1 + 4r^2 a^2) + 4r\dot{r}^2 a^2 - r\dot{\varphi}^2 + 2gra = 0 \,. \tag{2}$$

(c) Now use the Lagrange formalism to determine the equations of motion, Eqs. (1) and (2).

**You are not asked to solve the equations of motion.**

(2) **Numerical treatment of a particle in a circular cone** (3 Punkte + 1 Bonusp.)
In the lecture we derived the equations of motion of a particle moving on the inner surface of a circular cone (opening angle $\alpha = \pi/4$). In cylindrical coordinates $(r, \varphi, z)$, they read

$$2\dot{r}\dot{\varphi} + r\ddot{\varphi} = 0 \tag{3}$$
$$2\ddot{r} - r\dot{\varphi}^2 + g = 0 \,. \tag{4}$$

(a) Show that (3) leads to $r^2\dot{\varphi} = $ const.; we'll call this constant $l_z/m$ for reasons that will become obvious soon.

(b) Use (a) to eliminate $\dot{\varphi}$ from (4) and solve the remaining second-order equation for $r$ numerically by writing a Julia notebook (or other means).
Hint: you can find a template for a program (cone_skeleton) on adam.

(c) The naive numerical solution will give you $r(t)$. How can you easily modify (b) to obtain $\varphi(t)$?

Plot trajectories of $r$ as a function of $\varphi$. Describe and discuss the trajectories for different initial conditions. Find initial conditions that lead to circular orbits.

(d) Check that your solution conserves the energy $E = m\dot{r}^2 + l_z^2/(2mr^2) + mgr$.

Particularly nice graphics will get a bonus point.

(3) **Is the Lagrangian unique?** (2 Punkte)

Let $L(\mathbf{q}, \dot{\mathbf{q}}, t)$ be a Lagrangian of a system. The equations of motion of the system are then given by Lagrange's equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \ .$$

Show that

$$\tilde{L}(\mathbf{q}, \dot{\mathbf{q}}, t) = L(\mathbf{q}, \dot{\mathbf{q}}, t) + \frac{d}{dt}G(\mathbf{q}, t) \tag{5}$$

also fulfills Lagrange's equations.

(4) **Charge in an electromagnetic field** (2 Punkte)

The electromagnetic force on a particle with charge $q$ is

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

Show that $\mathbf{F}$ can be obtained from a generalized (velocity-dependent) potential $U(\mathbf{r}, \mathbf{v}) = q(\phi(\mathbf{r}) - \mathbf{v} \cdot \mathbf{A}(\mathbf{r}))$ by using

$$\mathbf{F} = -\nabla U + \frac{d}{dt}\frac{\partial}{\partial \mathbf{v}}U \ .$$

Hint: The electrostatic potential $\phi$ and the vector potential $\mathbf{A}$ are connected to the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ by $\mathbf{E} = -\nabla\phi - \dfrac{\partial \mathbf{A}}{\partial t}$ und $\mathbf{B} = \nabla \times \mathbf{A}$ .