Departement Physik, Universität Basel
Prof. C. Bruder (Zimmer 4.2, Tel.: 207 36 92, Christoph.Bruder@unibas.ch)

Quantenmechanik, Herbstsemester 2021

Blatt 8

Abgabe: 16.11.21, 12:00H (Treppenhaus 4. Stock)
Tutor: Ryan Tan, Zi. 4.10

(1) **Isotropic 3-dimensional harmonic oscillator** (3 Punkte)
Consider the isotropic 3-dimensional harmonic oscillator defined by the Hamiltonian

$$H = \frac{1}{2m} \sum_{i \in \{x,y,z\}} p_i^2 + \frac{1}{2} m\omega^2 \sum_{i \in \{x,y,z\}} x_i^2 .$$

As usual when dealing with spherically symmetric potentials we split the wavefunction in radial and angular parts and write

$$\Psi_{Elm} = \frac{u_{El}(r)}{r} Y_{lm}(\theta, \phi),$$

and obtain the Schrödinger equation for the radial part

$$\left[-\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + \frac{1}{2} m\omega^2 r^2 + \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2} - E \right] u(r) = 0,$$

where the $(E, l)-$dependence of $u(r)$ has been omitted.

(a) Using the ansatz wavefunction

$$u(r) = e^{-y^2/2} v(y), \qquad y = r\sqrt{m\omega/\hbar},$$

to derive a differential equation for $v(y)$.
What is the boundary condition on $v(y)$ for $y = 0$?
Result: $v''(y) - 2yv'(y) + [2\lambda - 1 - l(l+1)/y^2]v(y) = 0$ where $\lambda = E/(\hbar\omega)$.

(b) To further investigate the differential equation obtained in (a), we expand $v(y)$ as

$$v(y) = y^{l+1} \sum_{j=0}^{\infty} C_j y^j \; . \tag{1}$$

Explain the factor r^{l+1}. Derive a two-term recursion relation for the coefficients C_j, i.e., an equation relating C_j and C_{j+2}. Hint: show that $C_1 = 0$.

(c) Find the energy eigenvalues by using the two-term recursion relation and the requirement that the series (1) has to converge for $r \to \infty$.

(d) Compare the result from (c) with the energy quantization found when using second quantized operators, i.e., $H = \hbar\omega \sum_{i \in \{x,y,z\}} (a_i^\dagger a_i + \frac{1}{2})$, which was discussed in the lecture.
Show explicitly that both methods give the same (i) quantization of the energy, and (ii) degeneracy of the levels.

(2) **Hydrogen atom in $n = 2$, $l = 1$ state** (3 Punkte)

Assume that the electron in a hydrogen atom occupies the following combined spin and position eigenstate

$$\psi(r,\theta,\phi) = R_{21}(r)\left(\sqrt{\frac{1}{3}}Y_{10}(\theta,\phi)|\uparrow\rangle + \sqrt{\frac{2}{3}}Y_{11}(\theta,\phi)|\downarrow\rangle\right) .$$

(a) What are the possible measurement results if you measure the z-component of angular momentum L_z, and what is the probability of each?

(b) Same for the z-component of spin angular momentum S_z.

(c) Let $\mathbf{J} = \mathbf{L} + \mathbf{S}$ be the total angular momentum. If you measure J^2, what are the possible measurement results, and what is the probability of each?
Hint: Use the Clebsch-Gordan table.

(d) If you measure the position of the electron, what is the probability of finding it at r, θ, ϕ? Hint: $R_{21}(r) = \frac{1}{2\sqrt{6}}a^{-3/2}(\frac{r}{a})\exp(-r/(2a))$ where a is the Bohr radius.

(3) **Coupled spins** (2 Punkte + 2 Bonuspunkte)

Consider two coupled spin 1/2 particles in an external magnetic field $\mathbf{B} = (0, 0, B)$ along the z-direction, with the Hamiltonian

$$H = J\mathbf{S}_1 \cdot \mathbf{S}_2 - \frac{\mu}{\hbar}B(S_{1z} + S_{2z}) ,$$

where $J > 0$. Determine and discuss the energy eigenvalues and eigenstates. Plot the energies of the ground state and of the excited states as a function of the magnetic field B.

Hint: Consider $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$ and express the Hamiltonian in terms of S^2, S_1^2, and S_2^2.

Bonus points: Same for three spins arranged on the corners of a triangle. This situation is called "geometrically frustrated", can you imagine why? Use a computer for $B \neq 0$.

(4) **Variational method** (2 Punkte + 2 Bonuspunkte)

(a) Use the variational ansatz $\psi_\lambda(x) \propto \exp(-\frac{x^2}{2\lambda^2})$ to show that the one-dimensional potential well

$$V(x) = \begin{cases} -V_0 & |x| < a, \\ 0 & |x| \geq a \end{cases}$$

with $V_0 > 0$ has *at least* one bound state.
Hint: Show that it is possible to find a negative upper bound for the ground-state energy.
Bonus points: Show that the existence of a bound state is *not* guaranteed in the three-dimensional case. And what about the two-dimensional case?

(b) Find an upper bound for the ground-state energy of the Hamiltonian

$$H = \frac{p^2}{2m} + kx^4, \qquad k > 0 ,$$

by choosing an appropriate trial wavefunction.
Compare with the exact result

$$E_0 = 0.66798626\ldots\left(\frac{\hbar^4 k}{m^2}\right)^{1/3} .$$