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Quantenmechanik, Herbstsemester 2021

Blatt 5

Abgabe: 26.10.21, 12:00H (Treppenhaus 4. Stock)
Tutor: Julian Arnold, Zi. 4.48

(1) **Forced harmonic oscillator** (3 Punkte)

Consider a 1D harmonic oscillator that is in its ground state $|0\rangle_\mathrm{i}$ for $t \to -\infty$. We now apply an arbitrary time-dependent external force $F(t)$ with $F(t) = 0$ for $t \to \pm\infty$.

(a) Write down the Hamiltonian and express it terms of the initial lowering/raising operators a_i, $a_\mathrm{i}^\dagger$.

(b) Write down the Heisenberg equation of motion for $a_\mathrm{H}(t)$.

(c) Integrate the Heisenberg equation.

(d) Calculate the probability to find the oscillator in its excited state $|n\rangle_\mathrm{f}$ for $t \to \infty$.
Hint: Construct lowering/raising operators a_f, $a_\mathrm{f}^\dagger$ by looking at $a_\mathrm{H}(t \to \infty)$.
Then, use $|n\rangle_\mathrm{f} = \frac{1}{\sqrt{n!}}(a_\mathrm{f}^\dagger)^n|0\rangle_\mathrm{f}$ and the fact that the initial state is time-independent in the Heisenberg picture.

(2) **Binding δ-potential** (3 Punkte + 2 Bonuspunkte)

Consider a particle of mass m in the one-dimensional potential $V(x) = V_0\,\delta(x)$, with $V_0 < 0$.

(a) Find the bound state(s) (energy $E < 0$) of the particle in this potential. Sketch and interpret the wavefunction(s).
Hint: Either use the matching condition for wavefunctions in a δ-potential derived in the lecture, or go to momentum space.

(b) We now consider positive energies ($E > 0$). Make an ansatz for (unnormalized) scattering states. Derive the transmission amplitude $S(E)$ and the transmission probability $T(E)$ for a particle with energy E incident from the left.

(c) Bonus points: Re-derive the continuity equation for probability in the case of an imaginary potential. How do you interpret the role of imaginary potential in quantum mechanics? Next, consider an imaginary δ-potential $V(x) = \pm iV_0\,\delta(x)$ with $V_0 > 0$ real. Derive an expression for the absorption coefficient $\alpha(E) = 1 - T(E) - R(E)$, where $T(E)$ and $R(E)$ are the transmission and reflection probabilities for a particle with energy E incident from the left.
Remark: Non-hermitian Hamiltonians are a fashionable (and controversial) topic of current research.

(3) **More on the charged particle in a magnetic field** (4 Punkte + 1 Bonuspunkt)
We consider a particle (charge q) in a magnetic field $\mathbf{B}(\mathbf{r})$.

(a) Show that the (Heisenberg) equation of motion for the velocity $\mathbf{v}$ reads

$$\frac{d}{dt}\mathbf{v} = \frac{q}{m}(\mathbf{v} \times \mathbf{B}) + i\frac{q\hbar}{2m^2}\nabla \times \mathbf{B}\,. \tag{1}$$

Hint: Write the Hamiltonian in terms of velocities and use the commutation relation for the components of the velocity, $[v_k, v_l] = i\hbar\frac{q}{m^2}\epsilon_{kln}B_n$.
Interpret the terms on the right-hand side of Eq. (1).

(b) We now assume that $\mathbf{B} = (0, 0, B) = const.$ Solve Eq. (1) and show that the particle performs a cyclotron motion around a center (x_0, y_0) as in the classical case.

(c) Show that the *guiding center coordinates* x_0, y_0 are constants of motion but do not commute. Interpretation?

(d) Show that the square of the radius of the cyclotron motion has a sharp value in a (Landau) energy eigenstate and conclude that the radius of the stationary states grows like $\sqrt{n}$ for large Landau level number n.

(e) Bonus point: How is Eq. (1) modified if there is an additional electrical potential term $+q\phi(\mathbf{x}, t)$ in the Hamiltonian?
Hint: To ensure that the new expression is gauge-invariant you have to allow an explicit time dependence of $\mathbf{A}$.