

Quantenmechanik, Herbstsemester 2021

Blatt 12 (Zusatzpunkte)

Abgabe: 14.12.21, 12:00H (Treppenhaus 4. Stock)

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Schriftlicher Test: Dienstag, 21. Dezember 2021, 10.15 - 12 Uhr

Hilfsmittel: Ein beidseitig **hand**beschriebenes A4 Blatt.

(1) **Distinguishable (labeled) particles, fermions, bosons** (5 Bonuspunkte)

- (a) Two identical bosons are found to be in states $|\phi\rangle$ and $|\psi\rangle$. Write down the normalized state vector describing the system when $\langle\phi|\psi\rangle \neq 0$.
- (b) When an energy measurement is made on a system of three bosons in a box, the n values obtained were 3, 3, and 4. Write down a symmetrized, normalized state vector.
- (c) Consider three particles each of which can be in states ϕ_a , ϕ_b , and ϕ_c . Show that the total number of allowed, distinct configurations for this system is
 - i. 27 if they are labeled
 - ii. 10 if they are bosons
 - iii. 1 if they are fermions

(2) **Toy model for bunching/antibunching** (5 Bonuspunkte)

Consider two particles in two orthonormal states $\psi_a(x_1)$, $\psi_b(x_2)$.

- (a) Write down the 2-particle wave function for distinguishable particles, for indistinguishable bosons, and for indistinguishable fermions.
- (b) Calculate the expectation value $\langle(\Delta x)^2\rangle := \langle(x_1 - x_2)^2\rangle$ of the square of the distance between the two particles for all three cases and interpret your result.
- (c) Do this calculation for two particles in states ψ_n and ψ_m in an infinite square well.

Weitere Aufgaben für Unentwegte.
Abgabe bis 21.1.2022, Besprechung nach Vereinbarung

(3) **Attractive spherical δ -function shell** (10 Punkte)

Problem 3 on Blatt 11 studied the scattering amplitude and scattering phase shift of a repulsive spherical δ -function shell. The goal of the present problem is to analyze the very interesting case of an *attractive* shell that exhibits both bound states and so-called scattering resonances.

Investigate the bound states of the three-dimensional δ -shell potential

$$V(r) = -\alpha\delta(r - a).$$

It is useful to introduce the dimensionless variables $y := r/a$, $\xi := ka$, and $\beta := 2ma\alpha/\hbar^2$. It turns out that there is at most one bound state for each l .

- Determine the s-wave function. Show that a bound state exists only for $\beta > 1$.
- Show that there is at most one bound state corresponding to each l .
- Show for general l that the minimum strength of the potential for the existence of a bound state is $\beta = 2l + 1$.
- Calculate the scattering phases $\delta_l(k)$.
- Give the scattering cross section for s-waves.
- Determine the condition for the maxima of the s-wave scattering cross section.
- From here on, assume $\beta \gg \pi$. Determine the maxima for $ka \ll \beta$.
- Show that there are sharp and broad resonances. Show that the Breit-Wigner formula $\sigma \sim \Gamma^2/[(E - E_R)^2 + \Gamma^2]$ holds near the sharp resonances.
- Determine the poles of $e^{2i\delta_l} - 1$ on the negative real E -axis and interpret them.

(4) **Fractional quantum Hall effect** (5 Punkte)

The Laughlin wave function

$$\psi(z_1, z_2, \dots, z_N) = A \left[\prod_{j < k} (z_j - z_k)^q \right] \exp\left[-\frac{1}{2} \sum_k |z_k|^2\right]$$

is an approximate description of the ground state of N interacting electrons confined to two dimensions in a perpendicular magnetic field B . Here, q is a positive integer, and $z_j := \sqrt{\frac{eB}{2\hbar c}}(x_j + iy_j)$. Spin is not an issue here: in the ground state, all the electrons have spin down with respect to the direction of \mathbf{B} , and that is a trivially symmetric configuration.

- Show that ψ has the proper antisymmetry for fermions.
- For $q = 1$, ψ describes noninteracting particles (i.e., can be written as a single Slater determinant). Check this explicitly for $N = 3$. What single-particle states are occupied in this case?

- (c) For $q > 1$, ψ *cannot* be written as a single Slater determinant and describes interacting particles. It can, however, be written as a sum of Slater determinants. Show that, for $q = 3$ and $N = 2$, ψ can be written as a sum of two Slater determinants.

(5) **Shifted creation/annihilation operators** (5 Punkte)

Study the shift from the (bosonic or fermionic) operators \hat{a} , \hat{a}^\dagger to new operators $\hat{a}' = \hat{a} + \alpha$, $\hat{a}'^\dagger = \hat{a}^\dagger + \alpha^*$; here α is a complex number. Is this transformation unitary, i.e., does it preserve the fermionic/bosonic commutation relations? If so, give an explicit expression for the unitary operator.

Hint: $e^{\hat{A}\hat{B}}e^{-\hat{A}} = \hat{B} + \frac{1}{1!}[\hat{A}, \hat{B}] + \frac{1}{2!}[\hat{A}, [\hat{A}, \hat{B}]] + \dots$

(6) **Tight-binding model in second quantization** (5 Punkte)

A major part of solid-state physics deals with electrons in a periodic potential. As a simplified model we consider fermionic particles moving on a cubic lattice (lattice constant a). The kinetic energy is assumed to have tight-binding form

$$H = -t \sum_{\langle i,j \rangle \sigma} \left[c_{i\sigma}^\dagger c_{j\sigma} + c_{j\sigma}^\dagger c_{i\sigma} \right],$$

here, $\sum_{\langle i,j \rangle}$ is the sum over all nearest neighbors (such that each bond appears only once) and \sum_σ is the sum over the two spin directions.

- (a) Determine the band structure $\epsilon(\mathbf{k})$ for a d -dimensional cubic lattice ($d = 1, 2, 3$).
 (b) Draw the contours $\epsilon(\mathbf{k}) = \text{const.}$ in the (k_x, k_y) -plane for $d = 2$.

Hint: Diagonalize the Hamiltonian by a Fourier transform, $c_{j\sigma} = \frac{1}{\sqrt{N}} \sum_{\mathbf{k}} \exp(i\mathbf{k}\mathbf{r}_j) c_{\mathbf{k}\sigma}$, here, \mathbf{r}_j are the coordinates of the lattice sites; N is their total number.