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# Quantenmechanik, Herbstsemester 2021

## Blatt 10

Abgabe: 30.11.21, 12:00H (Treppenhaus 4. Stock)
Tutor: Gaomin Tang, Zi.: 4.48

(1) **Time-dependent two-level system; Rabi oscillations** (4 Punkte)
Consider the two-level system described by the Hamiltonian

$$H_0 = \epsilon_1 |1\rangle\langle 1| + \epsilon_2 |2\rangle\langle 2|$$

with $\epsilon_2 > \epsilon_1$ and define $\omega_{21} = (\epsilon_2 - \epsilon_1)/\hbar$.
The initial state of the system at time $t = 0$ is $|\psi(0)\rangle = |1\rangle$.

For $t > 0$, the system is subject to the time-dependent perturbation

$$V(t) = \gamma(e^{i\omega t}|1\rangle\langle 2| + e^{-i\omega t}|2\rangle\langle 1|).$$

(a) Identify the parameters $\epsilon_1$, $\epsilon_2$, $\gamma$, $\omega_{21}$, and $\omega$ if the two-level system is a spin 1/2 in a magnetic field.

(b) Use first-order time-dependent perturbation theory to calculate the probability $P_{12}(t)$ to find the system in state $|2\rangle$ at time $t$.

(c) [(c) and (d) are independent of (b)] We now want to solve the problem exactly to check the accuracy of the perturbative result. Use the ansatz

$$|\psi(t)\rangle = c_1(t)e^{-i\epsilon_1 t/\hbar}|1\rangle + c_2(t)e^{-i\epsilon_2 t/\hbar}|2\rangle$$

and derive differential equations for $c_1(t)$ and $c_2(t)$ from the Schrödinger equation.

(d) Integrate these differential equations for the initial condition $|\psi(0)\rangle = |1\rangle$ and show that the probability to find the system in state $|2\rangle$ at time $t$ is

$$P_{12}(t) = \frac{\gamma^2}{\gamma^2 + \hbar^2(\omega - \omega_{21})^2/4} \sin^2\left(\frac{t}{\hbar}\sqrt{\gamma^2 + \frac{\hbar^2(\omega - \omega_{21})^2}{4}}\right) .$$

Interpret this result, the **Rabi formula**. in particular, the limit $\omega \to \omega_{21}$.

(e) Compare the results in (b) and (d) and discuss carefully in which parameter range perturbation theory is accurate.

(2) **One-dimensional toy model for the photoelectric effect** (3 Punkte)
Consider an electron bound in an attractive $\delta$-function potential, $H_0 = \frac{-\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \alpha\delta(x)$. Calculate the probability per unit time of "ionization" if the electron is under the influence of a harmonically varying electric field, i.e., a perturbation $V(x,t) = -xeE_0 \cos\omega t$.

(a) Solve the problem assuming that the final states do not "see" the $\delta$-function potential (i.e., assume that the final states are plane waves).
Hint: Golden rule. The ground state of $H_0$ was found in problem 2 of Blatt 5: $\psi_0(x) = \sqrt{\kappa}e^{-\kappa|x|}$ where $\kappa = \frac{m\alpha}{\hbar^2}$; the ground-state energy is $\frac{-\hbar^2\kappa^2}{2m}$.

(b) Repeat (a) taking into account the influence of the $\delta$-function potential on the final states.

(3) **Wigner-Eckart theorem** (3 Punkte)

Electromagnetic quadrupole transitions in the hydrogen atom are described by matrix elements of the (spherical) quadrupole operators $Q_m^{(2)} \sim r^2 Y_{2m}$ that form a set of spherical tensor operators.

(a) Calculate the ratio $B/A$ of the following matrix elements; here, $|nlm\rangle$ are the eigenstates of the hydrogen atom:

$$A = \langle n'43|Q_2^{(2)}|n21\rangle \,, \\ B = \langle n'4,-2|Q_0^{(2)}|n2,-2\rangle \,.$$

(b) Calculate

$$C = \langle n'51|Q_2^{(2)}|n1,-1\rangle \,, \\ D = \langle n'31|Q_0^{(2)}|n1,-1\rangle \,.$$

(c) Consider the matrix element $\langle 4lm|z(x+iy)|n21\rangle$ where $x$,$y$,$z$ are Cartesian coordinates. Which values for $l$ and $m$ are allowed, i.e., lead to non-vanishing values?

(4) **Wigner function = supplement to Yiwen Chu's colloquium** (4 Bonuspunkte)

Knowing the density operator $\hat{\rho}$ of a particle is equivalent to knowing its density matrix $\rho(\mathbf{x},\mathbf{x}') = \langle \mathbf{x}|\hat{\rho}|\mathbf{x}'\rangle$ in the position representation (Hint: partition of unity).

We now consider a one-dimensional situation (the generalization is easy) and use $\rho(x,x')$ to define the *Wigner function*

$$f(r,p) = \frac{1}{2\pi\hbar}\int \mathrm{d}y \exp(ipy/\hbar)\rho(r+\frac{y}{2}, r-\frac{y}{2}) \,.$$

Show that $f(r,p)$ has the following properties:

(a) $f(r,p)$ is real.

(b) $\int \mathrm{d}p f(r,p)$ is the correct quantum-mechanical probability density in position space.

(c) $\int \mathrm{d}r f(r,p)$ is the correct quantum-mechanical probability density in momentum space.

(d) Hence $f(r,p)$ looks like a classical phase-space distribution that reproduces quantum mechanics, which appears to be a contradiction to everything we know (e.g., the uncertainty relation).
Calculate and plot $f(r,p)$ for the one-dimensional harmonic oscillator prepared in its $n$-th eigenstate for $n = 0,1,2$ to see what is the problem. Use a computer if necessary.