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Nanophysics — Fall 2020

Quantum Computation and Quantum Communication
Exercise 1

due Friday, December 4, 2020

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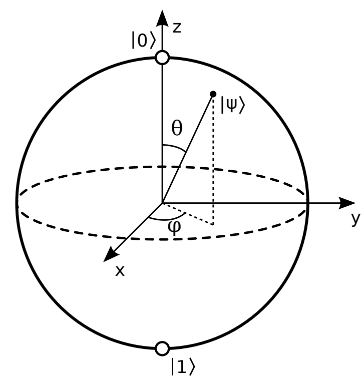
(1) **Qubit on a Bloch sphere**

A general state of a qubit $|\psi\rangle$ can be represented as a point on a Bloch sphere

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}\right)\mathrm{e}^{i\phi}|1\rangle,$$
$$\theta \in [0,\pi],\ \phi \in [0,2\pi)\,,$$

where the basis states $|0\rangle$ and $|1\rangle$ are the eigenvectors of the Pauli matrix $\sigma_z|0\rangle = |0\rangle$, $\sigma_z|1\rangle = -|1\rangle$ representing a spin pointing up or down in the z-direction.



Reminder: In the standard representation the Pauli matrices take the following form

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

The respective eigenvectors are

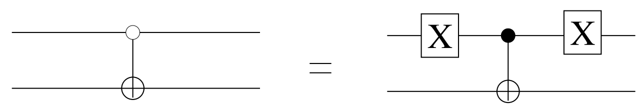
$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ |+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix},\ |-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix},\ \sigma_x|+\rangle = |+\rangle,\ \sigma_x|-\rangle = -|-\rangle\,.$$

(a) For a general state $|\psi\rangle$ from Eq. (1) compute the probability of finding the system in the state $|+\rangle$. Check your results against the known probabilities when $|\psi\rangle = \{|0\rangle, |+\rangle, |-\rangle\}$.

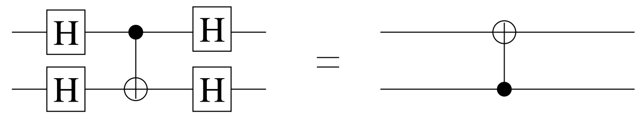
(b) Pauli matrices generate rotations on the Bloch sphere along the respective axes. So, for example, the rotation operator $R_y(\alpha)$ along the y axis with angle α is given as $R_y(\alpha) = \mathrm{e}^{-i\alpha\sigma_y/2}$. Find the explicit matrix form of $R_y(\alpha)$ and $R_z(\alpha)$. Show that $R_y(\pi/2)R_z(\pi) = -iH$ where H is the Hadamard gate.

(2) **Controlled NOT gate**

(a) In the lecture, we defined the CNOT gate to change the state of the target qubit if the control qubit is in the state $|1\rangle$. Now, we want to change the state of the target qubit *if the control qubit is in the state* $|0\rangle$. In a circuit diagram, such an operation is represented by a CNOT gate with an *empty* circle on the control qubit, as shown below on the left-hand side. Verify that:



(b) Show that one can swap the role of the control and the target qubit gate by applying four Hadamard gates $\hat{H}$:



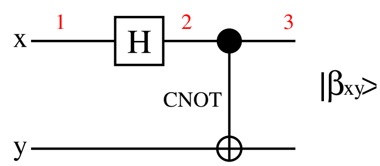
(3) **Bell states**

In the lecture, we defined the Bell states as

$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \ , \qquad |\beta_{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \ ,$$

$$|\beta_{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \ , \text{ and } \qquad |\beta_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \ .$$

(a) Verify that the circuit

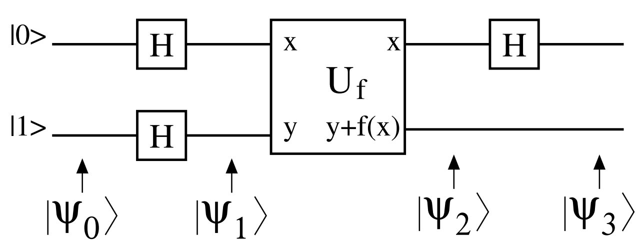


generates the Bell state $|\beta_{xy}\rangle$ if the input state is $|xy\rangle$ with $x, y \in \{0, 1\}$.

(b) Imagine that one of the qubits of a Bell state $|\beta_{xy}\rangle$ is sent to Alice and the other qubit of the same Bell state is sent to Bob (*i.e.*, Alice and Bob share a Bell state). Alice and Bob both apply a Hadamard gate $\hat{H}$ to their qubit. Show that two of the Bell states are interchanged, and two of the Bell states remain unchanged by this transformation (up to a global phase).

(4) **Deutsch algorithm**

Consider a binary classical function $f(x) : \{0, 1\} \to \{0, 1\}$. The Deutsch algorighm allows us to decide if $f(x)$ is balanced [*i.e.*, $f(0) \neq f(1)$] or constant [*i.e.*, $f(0) = f(1)$]. A quantum circuit implementing the Deutsch algorithm is:



$\hat{U}_f$ is a two-qubit gate that implements the transformation $\hat{U}_f|x, y\rangle = |x, y \oplus f(x)\rangle$, where the symbol $\oplus$ denotes addition modulo 2.

(a) Write down the state $|\psi_1\rangle$ obtained when the first two Hadamard gates $\hat{H}$ have acted on the input state $|\psi_0\rangle = |01\rangle$.

(b) Show that

$$|\psi_2\rangle = \begin{cases} \pm\frac{|0\rangle+|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle-|1\rangle}{\sqrt{2}} & \text{if } f(0) = f(1)\ , \\ \pm\frac{|0\rangle-|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle-|1\rangle}{\sqrt{2}} & \text{if } f(0) \neq f(1)\ . \end{cases}$$

(c) Show that the final state is

$$|\psi_3\rangle = \pm|f(0) \oplus f(1)\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}\ .$$