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Mechanik, Herbstsemester 2020

Blatt 11

Abgabe: 1.12.2020 **auf adam in den entsprechenden Ordner. Ein File pro Abgabe; der Filename muss Ihren Namen enthalten, sonst wird nicht korrigiert!**
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(1) **Hamilton-Jacobi equation for the central potential** (3 Punkte)
We want to solve the dynamics of a particle in a central potential

$$H = \frac{1}{2m}p_r^2 + \frac{1}{2mr^2}p_\phi^2 + V(r)$$

by using the Hamilton-Jacobi method.

(a) Write down the Hamilton-Jacobi equation and separate the time by the ansatz $S(r, \phi, \alpha_1, \alpha_2; t) = W(r, \phi, \alpha_1, \alpha_2) - \alpha_1 t$. Show that $\alpha_1 = E$ where E is the energy.

(b) Show that using the separation ansatz $W(r, \phi, E, \alpha_2) = W_1(r, E, \alpha_2) + W_2(\phi, E, \alpha_2)$ leads to the equation

$$\left(\frac{\partial W_2}{\partial \phi}\right)^2 = 2mr^2[E - V(r)] - r^2 \left(\frac{\partial W_1}{\partial r}\right)^2 .$$

Since the left and right side of this equation depend on different variables, they have to be equal to a constant (α_2). Solve the two equations and find the physical meaning of α_2.

(c) Use the relation $\partial S/\partial \alpha_i = \beta_i$ to find $t = t(r)$ and $\phi = \phi(r)$.

(2) **Phase portraits and Liouville's theorem** (5 Punkte)
Consider the dimensionless Hamiltonian $H(q, p) = \dfrac{p^2}{2} + V(q)$, where $V(q)$ is given by

(a) $V(q) = \dfrac{q^2}{2}$ (harmonic oscillator).

(b) $V(q) = \dfrac{q^2}{2} + q^4$ (anharmonic oscillator).

(c) $V(q) = 1 - \cos(q)$ (pendulum; q corresponds to the deflection angle).

For each of these examples, draw phase portraits by plotting phase-space orbits $(q(t), p(t))$ for a number of equidistant energies. Discuss the difference between open and closed orbits in (c). What happens for the initial condition $q = 0$, $p = 2$ in (c)?

Consider now the rectangular phase-space volume $-\frac{1}{2} \leq q \leq \frac{1}{2}, 1 \leq p \leq 3)$ at $t = 0$. Analyze its time evolution qualitatively by carefully sketching (or numerically calculating) its shape for different times (at least for a time $t \approx 1$ and for a time $t \gg 1$).

(3) **Poincaré sections** (2 Punkte + 5 Bonuspunkte)

The goal of this problem is to calculate and discuss Poincaré sections.

(a) As a warm-up, consider the anisotropic two-dimensional harmonic oscillator

$$H = \frac{1}{2m}(p_1^2 + p_2^2) + \frac{1}{2}m\omega_1^2 q_1^2 + \frac{1}{2}m\omega_2^2 q_2^2$$

with $\omega_2 = \sqrt{2}\omega_1$ and the initial condition $q_1(0) = q_2(0) = 0$, $p_1(0) = p_2(0) = \sqrt{mE/2}$ where E is the energy. Calculate and sketch the points at which the trajectory crosses the q_2, p_2-plane with $q_1 = 0$, $p_1 > 0$. Repeat this for the points at which the trajectory crosses the q_1, p_1-plane with $q_2 = 0$, $p_2 > 0$.

Hint: you don't really need a computer to solve (a).

(b) Bonus point: what happens if the two frequencies are commensurate, $\omega_2 = n\omega_1$ with integer n?

Now to a more interesting case **that exhibits chaos**: the double pendulum. The Hamiltonian was given in problem 3(e) of Blatt 9, and we choose the parameters $m_1 = m_2 = 1$, $g = 1$, and $a = 1$.

(b) Use Julia (or your favorite numerical method) to calculate the points at which the trajectory crosses the q_2, p_2-plane with $q_1 = 0$, $p_1 > 0$. Use the initial conditions $p_1(0) = 0.1$, $p_2(0) = 1.4$, $q_1(0) = 0.1$, $q_2(0) = 0$ and integrate the equations of motion in the time interval $[0, 3000]$.

Hint: a Julia notebook that solves this task for the Hénon-Heiles potential that was discussed in the lecture can be found in the notebook folder on adam.

(c) Repeat the procedure for the q_1, p_1-plane with $q_2 = 0$, $p_2 > 0$ and the initial conditions $p_1(0) = 0$, $p_2(0) = 0$, $q_1(0) = 0.1$, $q_2(0) = 0.2$.

(d) We now assume that the energy E is given. Find initial conditions that are compatible with E. Calculate the Poincaré sections in the q_1, p_1-plane with $q_2 = 0$, $p_2 > 0$ for the following energies: $E = -2.9$, $E = 0$, and $E = 10$. Try other possibilities!