Departement Physik, Universität Basel
Prof. C. Bruder (Zimmer 4.2, Tel.: 207 36 92, Christoph.Bruder@unibas.ch)

Mechanik, Herbstsemester 2020

Blatt 10

Abgabe: 24.11.2020 12:00H, **auf adam in den entsprechenden Ordner. Ein File pro Abgabe; der Filename muss Ihren Namen enthalten, sonst wird nicht korrigiert!**
Tutor: Ryan Tan, ryanguangting.tan@unibas.ch

(1) **Properties of the Poisson bracket** (5 Punkte)
The Poisson bracket of two observables (i.e., functions on phase space) $f(\mathbf{q}, \mathbf{p}; t)$ and $g(\mathbf{q}, \mathbf{p}; t)$ is defined as

$$\{f, g\} \equiv \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)$$

where n is the number of degrees of freedom.

(a) Show that the Poisson bracket has the following properties
   i. linearity $\{c_1 f + c_2 g, h\} = c_1\{f, h\} + c_2\{g, h\}$.
   ii. antisymmetry $\{f, g\} = -\{g, f\}$.
   iii. zero element $\{c, f\} = 0$ for $c = \text{const.}$
   iv. product rule $\{fg, h\} = f\{g, h\} + \{f, h\}g$.
   v. Bonus point: Jacobi identity $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$

(b) Assume that f and g are two constants of motion. Use the Jacobi identity to show that $\{f, g\}$ is also a constant of motion.

(c) Repeat (b) for the case that f and g are explicitly time-dependent.

(d) Write down the equations of motion of the one-dimensional harmonic oscillator using the Poisson bracket.

(e) Bonus point: Show that the components of the angular momentum vector $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ fulfill $\{L_x, L_y\} = L_z$ (and cyclic permutations).

(2) **Canonical transformations** (5 Punkte)
We consider a system with one degree of freedom ($f = 1$) and want to study canonical transformations from (q, p) to (Q, P) generated by $F_2(q, P, t)$ which leads to

$$Q = \frac{\partial F_2}{\partial P} \qquad\qquad p = \frac{\partial F_2}{\partial q}\,.$$

Construct suitable generators F_2 for the following cases:

(a) Identical transformation: $Q = q$, $P = p$.

(b) Galilei transformation, for which Q, P belong to a system moving with velocity v with respect to the original system: $q = Q + vt$, $p = P + mv$. Write down the new Hamiltonian $K(Q, P)$ that is related to the original Hamiltonian H by the relation $K = H + \frac{\partial F_2}{\partial t}$.

(c) A general point transformation from q to the new coordinate Q, i.e., $Q = g(q, t)$. Construct a generator F_2 such that the new momentum P is *proportional* to the original momentum p. Show that the transformation rule for the momentum can also be obtained from the fact that the Lagrangian is invariant under a point transformation: $L(q, \dot{q}; t) = \tilde{L}(Q, \dot{Q}; t)$.