Departement Physik, Universität Basel
Prof. C. Bruder (Zimmer 4.2, Tel.: 207 36 92, Christoph.Bruder@unibas.ch)

Mechanik, Herbstsemester 2020

Blatt 0

Besprechung: 16.9.2020; 8:15 – 10:00 (wird in der Übung gerechnet, keine Abgabe!)
Tutor: Frank Schäfer, frank.schaefer@unibas.ch

(1) **Free fall of a ball: numerically solving ordinary differential equations (ODEs)**

Consider a ball (radius 0, we are physicists...) in the gravitational field of the Earth. If we assume the field to be homogeneous, the differential equation which describes the free fall of the ball reads

$$m\frac{\mathrm{d}^2h}{\mathrm{d}t^2} = -mg\,. \tag{1}$$

Here, $h(t)$ denotes its height, m its mass, and g is the gravitational acceleration.

We assume that at time $t = 0$, the initial value of the height h is $h(0) = h_0$, and the initial value of the velocity $v(t) \equiv \dot{h}(t) \equiv \frac{\mathrm{d}h}{\mathrm{d}t}(t)$ is $v(0) = v_0$.

In the following, we will describe a numerical procedure that can be applied to arbitrary ODE for which the initial values are given.

(a) Equation (1) is a second-order ODE. Show that any explicit ODE of order n can be reduced to n differential equations of order 1,

$$\dot{\mathbf{u}}(t) = \mathbf{f}(t, \mathbf{u})\,, \qquad \mathbf{u}(t_0) = \mathbf{u}_0\,, \tag{2}$$

here, $\mathbf{u}$, $\mathbf{f}$, and $\mathbf{u}_0$ are n-dimensional vectors.

Do this explicitly for Eq. (1). Solution:

$$\mathbf{u}(t) = \begin{pmatrix} h(t) \\ v(t) \end{pmatrix}\;; \qquad \mathbf{f}(t, \mathbf{u}) = \begin{pmatrix} v(t) \\ -g \end{pmatrix}\;; \qquad \mathbf{u}_0 = \begin{pmatrix} h_0 \\ v_0 \end{pmatrix}\,. \tag{3}$$

(b) The explicit Euler scheme

$$\mathbf{u}_{k+1} = \mathbf{u}_k + \Delta t \mathbf{f}(t_k, u_k) \tag{4}$$

gives an approximate solution $\mathbf{u}(t_j) = \mathbf{u}_j$ to the initial value problem defined by Eq. (2) in the interval $[0, T]$. The t_j, $j = 0, 1, \ldots, N$ are a discretization of the time interval; e.g., $t_j = t_0 + j \cdot \Delta t$ and $\Delta t = T/N$.

Assuming the initial values $h_0 = 50$ m, $v_0 = 0$, solve the differential equation (3) numerically by applying the explicit Euler scheme for $T = 10$ s and $N = 100$.

(c) Compare with the exact solution of (1) that you know from Physics I:

$$h(t) = h_0 + v_0 t - \frac{1}{2}gt^2\,. \tag{5}$$

(2) **Fall time: using the Newton scheme to solve non-linear equations**

(a) Calculate the time $t^* > 0$ at which the ball hits the ground $h = 0$. Solution:

$$h(t^*) = 0 \quad \Leftrightarrow \quad t^* = \frac{-v_0 + \sqrt{v_0^2 + 2gh_0}}{g} \tag{6}$$

(b) Now determine the solution to $h(t^*) = 0$ numerically by using the Newton scheme: to solve the nonlinear equation $G(x) = 0$, start from an initial guess x_0 and construct a sequence x_n, $n = 1, 2, \ldots$ according to

$$x_{n+1} = x_n - \frac{G(x_n)}{G'(x_n)} \,. \tag{7}$$

(c) Compare with the exact solution.

(3) **Bonus: Bouncing ball: event handling**

To describe the situation that the ball is bouncing off the ground ($h = 0$), we need to reverse the ball's velocity when it hits the ground. Implement this process in your numerical solution of the differential equation.