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Quantenmechanik, Herbstsemester 2019

Blatt 8

Abgabe: 12.11.19, 12:00H (Treppenhaus 4. Stock)
Tutor: Gaomin Tang, Zi.: 4.16

(1) **Clebsch-Gordan coefficients** (2 Punkte)

(a) Consider a system composed of a spin-3/2 particle and a spin-1 particle with spin z-components $S_{1z} = +1/2$ and $S_{2z} = 0$. What are the possible measurement results for a measurement of $\mathbf{S}^2$ and of S_z, where $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$ is the total spin of the system? What are the probabilities for each of these possible measurement results?

(b) Consider now the state of two coupled particles, again a spin-3/2 and a spin-1, with the total spin $S = 5/2$ and with $S_z = -1/2$. What are the possible measurement results for a measurement of S_{1z} and of S_{2z}? What are the probabilities for each of these possible results?

(2) **Two coupled spins** (2 Punkte)
Consider two coupled spin 1/2 particles in an external magnetic field $\mathbf{B} = (0, 0, B)$ along the z-direction, with the Hamiltonian

$$H = J\mathbf{S}_1 \cdot \mathbf{S}_2 - \frac{\mu}{\hbar} B(S_{1z} + S_{2z}) ,$$

where $J > 0$. Determine, plot, and discuss the spectrum of the system, i.e., the energies of the ground state and of the excited states, as a function of the magnetic field B.

Hint: Consider $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$ and express the Hamiltonian in terms of $\mathbf{S}^2$, $\mathbf{S}_1^2$, and $\mathbf{S}_2^2$.

(3) **Square of a spin** (3 Punkte)
Let $\mathbf{S}$ a spin operator with spin s. Since $\mathbf{S}^2|s, m\rangle = \hbar^2 s(s + 1)|s, m\rangle$, the spin matrices for a given s fulfill the matrix equation

$$S_x^2 + S_y^2 + S_z^2 = \hbar^2 s(s + 1)\mathbb{1} . \tag{1}$$

For $s = 1/2$, this is trivially fulfilled since $S_x^2 = S_y^2 = S_z^2 = \hbar^2\mathbb{1}/4$.

(a) Consider now a particle with spin $s = 1$. Show that S_x^2, S_y^2, and S_z^2 commute and find their joint eigenbasis. Write S_x, S_y, and S_z in this new basis.

(b) Using the joint eigenbasis found in (a), study how the general relation (1) holds.

(c) * Is a similar explanation possible for $s = 3/2$? Consider the eigenvalues of S_i^2 for $i = x, y, z$ and show that the eigenvalues on the left-hand side of relation (1) cannot add up to the eigenvalue on the right. Conclude (without calculation) that S_x^2, S_y^2, and S_z^2 cannot commute for $s = 3/2$.
(For an interesting comment on the significance of this result on the interpretation of quantum mechanics see Ballentine p. 175).

(4) **Variational method** (3 Punkte)

(a) Use the variational ansatz $\psi_\lambda(x) \propto \exp(-\frac{x^2}{2\lambda^2})$ to show that the one-dimensional potential well

$$V(x) = \begin{cases} -V_0 & |x| < a, \\ 0 & |x| \geq a \end{cases}$$

with $V_0 > 0$ has *at least* one bound state.
Hint: Show that it is possible to find a negative upper bound for the ground-state energy.
* Is this true in the two-dimensional or three-dimensional case?

(b) Find an upper bound for the ground-state energy of the Hamiltonian

$$H = \frac{p^2}{2m} + kx^4, \qquad k > 0\,,$$

e.g., using the variational wavefunction

$$\psi_\lambda(x) \propto \exp(-\frac{x^2}{2\lambda^2})\,. \tag{2}$$

Compare with the exact result

$$E_0 = 0.66798626\ldots \left(\frac{\hbar^4 k}{m^2}\right)^{1/3}.$$

* Can you find a better variational wavefunction than (2)?