Departement Physik, Universität Basel
Prof. C. Bruder (Zimmer 4.2, Tel.: 207 36 92, Christoph.Bruder@unibas.ch)

Quantenmechanik, Herbstsemester 2019

Blatt 6

Abgabe: 29.10.19, 12:00H (Treppenhaus 4. Stock)
Tutor: Frank Schäfer, Zi. 4.13

(1) **Forced harmonic oscillator** (3 Punkte)
Consider a 1-dim harmonic oscillator that is in its ground state for $t \to -\infty$. We now apply an arbitrary time-dependent external force $F(t)$ with $F(t) \to 0$ for $t \to \pm\infty$.

(a) Write down the Hamiltonian and the Heisenberg equation of motion for $a_H(t)$.

(b) Integrate the Heisenberg equation. Calculate the probability to find the oscillator in its excited state $|n\rangle_{\mathrm{f}}$ for $t \to \infty$.
Hint: $|n\rangle_{\mathrm{f}}$ are the eigenstates of $a_H^\dagger(t) a_H(t)$ for $t \to \infty$, and they differ from $|n\rangle_{\mathrm{i}}$, the eigenstates of $a_H^\dagger(t) a_H(t)$ for $t \to -\infty$.

(2) **Components of the angular momentum operator** (2 Punkte)
Using the commutation relation $[x_j, p_k] = i\hbar\delta_{jk}$, show the following relations for the orbital angular momentum operator $\mathbf{L} = (L_x, L_y, L_z)$; $\mathbf{n}$ is a real vector.

(a) $[\mathbf{n} \cdot \mathbf{L}, \mathbf{r}] = i\hbar \mathbf{r} \times \mathbf{n}$

(b) $[\mathbf{n} \cdot \mathbf{L}, \mathbf{p}] = i\hbar \mathbf{p} \times \mathbf{n}$

(c) $\mathbf{L} \times \mathbf{L} = i\hbar \mathbf{L}$

(3) **Can l be half-integer for orbital angular momenta?** (3 Punkte)
In this problem, we will prove that the eigenvalues m of the angular momentum operator $L_z/\hbar$ must be integer. Therefore, the orbital angular momentum quantum number l can take only integer values.

(a) Express the operator L_z in terms of the creation and destruction operators, $a_i^\dagger$ and a_i $(i = 1, 2, 3)$ by using the transformations

$$x_i = \sqrt{\frac{\hbar}{2m\omega}}(a_i + a_i^\dagger)$$
$$p_i = -i\sqrt{\frac{\hbar m\omega}{2}}(a_i - a_i^\dagger) \, .$$

(b) By introducing new operators b_1, b_2 (and their hermitian conjugates) that are linear combinations of the a_i's, show that L_z can be written in the form

$$L_z = \hbar(b_2^\dagger b_2 - b_1^\dagger b_1) \, ,$$

where the operators b_1 and b_2 satisfy the commutation relations $[b_1, b_1^\dagger] = [b_2, b_2^\dagger] = 1$, $[b_1, b_1] = [b_1^\dagger, b_1^\dagger] = 0$, and so on.

(c) Argue that the eigenvalues of L_z should be an integer multiplied by $\hbar$ and consequently that the orbital angular momentum l should be integer.

(4) **Kronig-Penney-Model for $V_0 < 0$** (2 Punkte + 3 Extra-Punkte)
In the lecture we discussed the Kronig-Penney-Model

$$V(x) = V_0 \sum_{n=-\infty}^{\infty} \delta(x - na)\,.$$

Solving the equation

$$\cos(ka) = \cos(qa) + \frac{mV_0 a}{\hbar^2} \frac{\sin(qa)}{qa} \tag{1}$$

graphically for $V_0 > 0$, we obtained the allowed and forbidden values of q which resulted in energy bands and energy gaps in $\epsilon_k = \hbar^2 q^2/(2m)$.

Consider now the case $V_0 < 0$.

(a) Sketch the right-hand side of Eq. (1) as a function of qa and solve the equation graphically or with a computer.

(b) Sketch the lowest energy band ϵ_k.

(c) On top of that there are also solutions to the Schrödinger equation with negative energy eigenvalues. What does this imply for q? Solve Eq. (1) for this case graphically or with a computer and sketch the part of the lowest energy band that originates from this solution. Interpret your result.
Calculate and plot the first five energy bands numerically.

(5) **Imaginary $\boldsymbol{\delta}$-potential** (small research problem...)
Repeat problem 2 from Blatt 5 for the potential $V(x) = \pm i V_0\, \delta(x)$ with $V_0 > 0$ real. Note: this Hamiltonian is non-hermitian but has real eigenvalues. Such Hamiltonians are a fashionable (and controversial) topic of current research.