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Quantenmechanik, Herbstsemester 2019

Blatt 5

Abgabe: 22.10.19, 12:00H (Treppenhaus 4. Stock)
Tutor: Frank Schäfer, Zi. 4.13

(1) **Gauge transformation in quantum mechanics** (3 Punkte)
The Schrödinger equation of a charged particle with charge q

$$i\hbar\partial_t\Psi(\mathbf{x},t) = \frac{1}{2m}\left(\hat{\mathbf{p}} - q\mathbf{A}\right)^2\Psi(\mathbf{x},t) + q\Phi\Psi(\mathbf{x},t)\,.$$

contains the vector potential $\mathbf{A}$ and the scalar potential Φ (that will in general depend on $\mathbf{x}$ and t), but not the fields $\mathbf{E} = -\nabla\Phi - \partial_t\mathbf{A}$ and $\mathbf{B} = \nabla\times\mathbf{A}$. A gauge transformation to new potentials $\mathbf{A}'$ und Φ' with

$$\mathbf{A}' \;=\; \mathbf{A} + \nabla\chi\,; \qquad \Phi' = \Phi - \partial_t\chi\,,$$

where $\chi(\mathbf{x},t)$ is an arbitrary smooth function, will not change the fields.

(a) Prove the following operator equation (you should think of both sides of the equation as operators that act on a wave function standing on their right):

$$e^{f(y)}\frac{\partial}{\partial y} = \left(\frac{\partial}{\partial y} - \frac{\partial f}{\partial y}\right)e^{f(y)}\,.$$

(b) Use (a) to show that the gauge-transformed wave function defined by

$$\Psi'(\mathbf{x},t) := \exp(iq\chi/\hbar)\Psi(\mathbf{x},t)$$

solves the modified Schrödinger equation that contains the transformed potential $\mathbf{A}'$ and Φ'.

(2) **Binding $\boldsymbol{\delta}$-potential** (3 Punkte)
Consider a particle of mass m in the one-dimensional potential $V(x) = V_0\,\delta(x)$, with $V_0 < 0$.

(a) Find the bound state(s) (energy $E < 0$) of the particle in this potential. Sketch and interpret the wavefunctions.
Hint: either use the matching condition for wavefunctions in a δ-potential derived in the lecture, or go to momentum space.

(b) We now consider positive energies ($E > 0$). Make an ansatz for (unnormalized) scattering states. Derive the transmission amplitude $S(E)$ and the transmission probability $T(E)$ for a particle with energy E incident from the left.

(c) * Determine the poles of $S(E)$ and discuss their physical significance.

(3) **More on the charged particle in a magnetic field** (4 Punkte)

We consider a particle (charge q) in a magnetic field $\mathbf{B}(\mathbf{r})$.

(a) Show that the (Heisenberg) equation of motion for the velocity $\mathbf{v}$ reads

$$\frac{d}{dt}\mathbf{v} = \frac{q}{m}(\mathbf{v} \times \mathbf{B}) + i\frac{q\hbar}{2m^2}\nabla \times \mathbf{B}\,. \tag{1}$$

Hint: Write the Hamiltonian in terms of velocities and use the commutation relation for the components of the velocity, $[v_k, v_l] = i\hbar\frac{q}{m^2}\epsilon_{kln}B_n$.
Interpret the terms on the right-hand side of Eq. (1).

(b) We now assume that $\mathbf{B} = (0, 0, B) = const$. Solve Eq. (1) and show that the particle performs a cyclotron motion around a center (x_0, y_0) as in the classical case.

(c) Show that the *guiding center coordinates* x_0, y_0 are constants of motion but do not commute. Interpretation?

(d) Show that the square of the radius of the cyclotron motion has a sharp value in a (Landau) energy eigenstate and conclude that the radius of the stationary states grows like $\sqrt{n}$ for large Landau level number n.

(e) * How is Eq. (1) modified if there is an additional electrical potential term $+q\phi(\mathbf{x}, t)$ in the Hamiltonian?
Hint: you have to allow an explicit time dependence of $\mathbf{A}$ to be gauge-invariant.