**Departement Physik, Universität Basel**
**Prof. C. Bruder (Zimmer 4.2, Tel.: 207 36 92, Christoph.Bruder@unibas.ch)**

# Quantenmechanik, Herbstsemester 2019

**Blatt 3**

Abgabe: 8.10.19, 12:00H (Treppenhaus 4. Stock)
Tutor: Michal Kloc, Zi.: 4.10

(1) **State determination** (2 Punkte)

Measurements on a system of two spin 1/2 particles yield the following expectation values:

$$\left\langle S_z^{(1)} \right\rangle = \left\langle S_z^{(2)} \right\rangle = 0 \quad \text{and} \quad \left\langle S_z^{(1)} \otimes S_z^{(2)} \right\rangle = \frac{\hbar^2}{4},$$

where $S_z = \frac{\hbar}{2}\sigma_z$.

(a) Construct a *pure* state consistent with the given data, or prove that none exists.

(b) Construct a *mixed* state consistent with the given data, or prove that none exists.

(2) **Reduced density operator** (3 Punkte)

Consider a system of two spins 1/2 in the state $|\psi_\alpha\rangle = \cos(\alpha)|\uparrow\downarrow\rangle + \sin(\alpha)|\downarrow\uparrow\rangle$.

(a) Write down the density operator $\rho$ that describes this system.

(b) Calculate the reduced density operator $\rho^{(1)}$ of subsystem 1 (i.e., the first spin). Does it represent a pure or a mixed state? What is the expectation value of $S_z^{(1)}$ and $S_x^{(1)}$ if only the first spin is measured. Interpret your results.

(c) For a joint measurement of $S_z^{(1)}$ and $S_x^{(2)}$ of both spins, calculate the probability to measure $+\hbar/2$ for spin 1 and $+\hbar/2$ for spin 2.

(3) **Commutators of components of x and p** (2 Punkte)

Let $F$ and $G$ be analytic functions of $\mathbf{p}$ and $\mathbf{x}$, respectively.
Show (e.g., by induction) that

(a) $[x_j, F(\mathbf{p})] = i\hbar \dfrac{\partial F}{\partial p_j}(\mathbf{p})$

(b) $[p_j, G(\mathbf{x})] = -i\hbar \dfrac{\partial G}{\partial x_j}(\mathbf{x})$

(4) **Time evolution** (3 Punkte)

Consider a spin 1/2 particle. At time $t = 0$, the system is prepared in the state

$$|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle_z + e^{i\phi}|\downarrow\rangle_z\Big) .$$

The Hamiltonian of the system is given by $H = -g\frac{\mu_B}{\hbar}\mathbf{B}\cdot\mathbf{S}$, with $\mathbf{B} = B_0\mathbf{e}_z$ and $\mathbf{S} = \frac{\hbar}{2}(\sigma_x, \sigma_y, \sigma_z)$.

(a) Write down a formal expression for the time evolution operator $U(t)$ and evaluate it explicitly. Calculate $|\psi(t)\rangle$ for $t > 0$.
*Hint*: Show that $\exp(i\alpha\sigma_z) = \mathbb{1}\cos\alpha + i\sigma_z\sin\alpha$.

(b) What is the probability that a measurement of $S_x$ done at time $t > 0$ gives $+\hbar/2$? What is the probability that a measurement of $S_z$ done at time $t > 0$ gives $-\hbar/2$?

(c) Calculate $\langle S_x(t)\rangle$, $\langle S_y(t)\rangle$, and $\langle S_z(t)\rangle$. Interpretation?

(5) **Zusatzaufgabe: Generalized uncertainty relation** (5 Extra-Punkte)

Assume that a system is described by the density operator $\hat{\rho}$. We consider two observables $A$, $B$ and define $A_0 := A - \langle A\rangle$ where $\langle A\rangle := \mathrm{Tr}(\hat{\rho}A)$ and the same for $B$. Their variances are defined as $\Delta_A^2 = \langle A_0^2\rangle = \mathrm{Tr}(\hat{\rho}A_0^2)$ and the same for $B$.

(a) Prove the generalized uncertainty relation:

$$\Delta_A\Delta_B \geq \frac{1}{2}\sqrt{\langle i[A_0, B_0]\rangle^2 + \langle\{A_0, B_0\}\rangle^2} . \tag{1}$$

Hint: Define $T := A_0 + i\omega B_0$ and find $\omega \in \mathbb{C}$ such that $\langle TT^\dagger\rangle$ is minimized.

(b) Show that for pure states, (1) reduces to the relation found in the lecture.

(c) Assume that the system is described by a completely mixed state with $\hat{\rho} \sim \mathbb{1}$. Apply (1) to this case. Apparently, there is a lower bound to the product of the two variances, regardless of whether $A$ and $B$ commute or not. Can you interpret this result?!