Departement Physik, Universität Basel
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Quantenmechanik, Herbstsemester 2019

Blatt 2

Abgabe: 1.10.19, 12:00H (Treppenhaus 4. Stock)
Tutor: Frank Schäfer, Zi.: 4.13

Schriftlicher Test: Dienstag, 17. Dezember 2019, 10.15 - 12 Uhr
Hilfsmittel: Ein **hand**beschriebenes A4 Blatt.

Mündliche Vorlesungprüfung: Freitag, 31. Januar 2020

(1) **Observables and hermitian operators** (1 Punkt)

Postulate 2 tells us that physical quantities correspond to hermitian operators, and this is good because their eigenvalues are real and can be interpreted as measurement results, see Postulate 3.
But why not simply require that the operators have real eigenvalues?
Consider the following non-hermitian matrix

$$M = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

(a) Show that M has real eigenvalues, but its eigenvectors do not form a complete set.

(b) Find a state $|v\rangle$ such that $\langle v|M|v\rangle$ is complex. Why is this a problem?

(2) **Different mixtures, same density operator** (3 Punkte)

A spin-$\frac{1}{2}$ particle is prepared in the following mixed state:

$$\begin{aligned} 50\% \quad & |\uparrow\rangle_z \,, \\ 50\% \quad & |\downarrow\rangle_x \,. \end{aligned}$$

Write down the density operator. Find a mixture of two orthogonal states that is described by the same density operator. Interpretation?
Hint: Either guess, or try the general ansatz $|\psi\rangle = \cos\frac{\theta}{2}|\uparrow\rangle + e^{i\phi}\sin\frac{\theta}{2}|\downarrow\rangle$ and write down a general mixture of $|\psi\rangle$ and its orthogonal partner.

(3) **Position and momentum eigenstates** (3 Punkte)

Let $|x\rangle$ be the eigenstates of the (1-dimensional) position operator $\hat{x}$ and $|p\rangle$ the eigenstates of the (1-dimensional) momentum operator $\hat{p}$.

Let $|\psi\rangle$ be an arbitrary normalized state. Prove the following relations. Write down all the intermediate steps and explain carefully what you do.

(a) $\langle p|\psi\rangle = \dfrac{1}{\sqrt{2\pi\hbar}} \displaystyle\int \mathrm{d}x \exp(-ipx/\hbar)\langle x|\psi\rangle$. What is the physical meaning of $\langle p|\psi\rangle$?

(b) $\int \mathrm{d}p \, |\langle p|\psi\rangle|^2 = 1$

(c) $\langle p|\hat{x}|x\rangle = \dfrac{1}{\sqrt{2\pi\hbar}} x \exp(-ipx/\hbar)$

(d) $\langle p|\hat{p}|\psi\rangle = p \, \langle p|\psi\rangle$

(e) $\langle x|\hat{p}^2|\psi\rangle = -\hbar^2 \dfrac{\partial^2}{\partial x^2} \langle x|\psi\rangle$

(f) $\langle p|\hat{x}|\psi\rangle = i\hbar \dfrac{\partial}{\partial p} \langle p|\psi\rangle$

(4) **State reduction after measurement** (3 Punkte)

(a) Consider a quantum system with a 3-dimensional state space. We would like to measure the physical quantity $\mathcal{A}$. The corresponding operator A has eigenvalues λ_n and eigenvectors $|e_n\rangle$: $A\,|e_n\rangle = \lambda_n\,|e_n\rangle$, $n = 1, 2, 3$. In addition, we assume that $\lambda_1 = \lambda_2 \equiv \lambda$, i.e., two of the eigenvalues are degenerate.

The system is prepared in state $|\psi\rangle = -\dfrac{i}{\sqrt{2}}\,|e_1\rangle + \dfrac{1}{2}\,|e_2\rangle - \dfrac{i}{2}\,|e_3\rangle$.

A measurement of $\mathcal{A}$ gives the result λ. Use Postulate P5 to calculate the (normalized) state of the system immediately after the measurement.

(b) Assume that you perform the same measurement but do not record the measurement result. What is the state/density operator after the measurement?

(c) Assume that you *do not know* the state of the system considered in (a).
A measurement of $\mathcal{A}$ gives the result λ. What can you tell about the (normalized) state after the measurement?

(d) Assume that a system is described by the density operator $\hat{\rho}$. Is it possible that a measurement leads to a mixed density operator $\hat{\rho}_{\text{red}}$? Give your reasons.