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Quantenmechanik, Herbstsemester 2019

Blatt 10

Abgabe: 26.11.19, 12:00H (Treppenhaus 4. Stock)
Tutor: Michal Kloc, Zi.: 4.10

(1) **Two spins with time-dependent coupling** (3 Punkte)

Consider two coupled spin 1/2 particles with a time-dependent coupling constant $J(t)$ which approaches zero for $t \to \pm\infty$ with $\alpha := \int_{-\infty}^{\infty} \mathrm{d}t J(t)$ finite. The Hamiltonian is

$$H(t) = J(t)\mathbf{S}_1 \cdot \mathbf{S}_2 \,.$$

Assume that the system is prepared in the state $|\psi(t \to -\infty)\rangle = |{\uparrow\downarrow}\rangle_z := |{\uparrow}\rangle_z^{(1)} \, |{\downarrow}\rangle_z^{(2)}$.

(a) Does H commute with itself at different times? Write down an expression for the time evolution operator $U(t, -\infty)$.

(b) Calculate the (exact) state of the system for $t \to +\infty$.

(c) What is the probability to find the system in the state $|{\downarrow\uparrow}\rangle_z$ for $t \to +\infty$?

(d) **[independent of (a) – (c)]**
Repeat (c) using first-order time-dependent perturbation theory, i.e., calculate the probability $P_{|\uparrow\downarrow\rangle\to|\downarrow\uparrow\rangle}(t = +\infty)$.
Compare with the exact result.

(2) **One-dimensional toy model for the photoelectric effect** (3 Punkte)

Consider an electron bound in an attractive δ-function potential, $H_0 = \frac{-\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \alpha\delta(x)$. Calculate the probability per unit time of "ionization" if the electron is under the influence of a harmonically varying electric field, i.e., a perturbation $V(x,t) = -xeE_0 \cos\omega t$.

(a) Solve the problem assuming that the final states do not "see" the δ-function potential.
Hint: Use the Golden rule. The ground state of H_0 was found in problem 2 of Blatt 5:
$\psi_0(x) = \sqrt{\kappa}e^{-\kappa|x|}$ where $\kappa = \frac{m\alpha}{\hbar^2}$; the ground-state energy is $\frac{-\hbar^2\kappa^2}{2m}$.

(b) Repeat (a) taking into account the influence of the δ-function potential on the final states.

(3) **Time-dependent two-level system; Rabi oscillations** (4 Punkte)

Consider the two-level system described by the Hamiltonian

$$H_0 = \epsilon_1 |1\rangle\langle 1| + \epsilon_2 |2\rangle\langle 2|$$

and define $\omega_{21} = (\epsilon_2 - \epsilon_1)/\hbar$. The initial state of the system at time $t = 0$ is $|\psi(0)\rangle = |1\rangle$. For $t > 0$, the system is subject to the time-dependent perturbation

$$V(t) = \gamma(e^{i\omega t}|1\rangle\langle 2| + e^{-i\omega t}|2\rangle\langle 1|).$$

(a) Identify the parameters ϵ_1, ϵ_2, γ, ω_{21}, and ω if the two-level system is a spin 1/2 in a magnetic field.

(b) Use first-order time-dependent perturbation theory to calculate the probability $P_{12}(t)$ to find the system in state $|2\rangle$ at time t.

(c) [(c) and (d) are independent of (b)] We now want to solve the problem exactly to check the accuracy of the perturbative result. Use the ansatz

$$|\psi(t)\rangle = c_1(t) e^{-i\epsilon_1 t/\hbar}|1\rangle + c_2(t) e^{-i\epsilon_2 t/\hbar}|2\rangle$$

and derive differential equations for $c_1(t)$ and $c_2(t)$ from the Schrödinger equation.

(d) Integrate these differential equations for the initial condition $|\psi(0)\rangle = |1\rangle$ and show that the probability to find the system in state $|2\rangle$ at time t is

$$P_{12}(t) = \frac{\gamma^2}{\gamma^2 + \hbar^2(\omega - \omega_{21})^2/4} \sin^2\left(\frac{t}{\hbar}\sqrt{\gamma^2 + \frac{\hbar^2(\omega - \omega_{21})^2}{4}}\right).$$

Interpret this result, the **Rabi formula**. in particular, the limit $\omega \to \omega_{21}$.

(e) Compare the results in (b) and (d) and discuss carefully in which parameter range perturbation theory is accurate.