Department of Physics, University of Basel
Martin Koppenhöfer (room 4.16, martin.koppenhoefer@unibas.ch)
Christian Scheller (room 1.10)

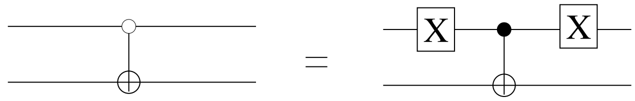
Nanophysics — Fall 2019

Quantum Computation and Quantum Communication
Exercise 1

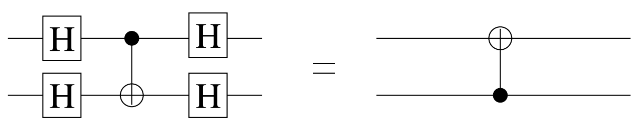
due Friday, December 6, 2019

(1) **Controlled NOT gate**

(a) In the lecture, we defined the CNOT gate to change the state of the target qubit if the control qubit is in the state $|1\rangle$. Now, we want to change the state of the target qubit *if the control qubit is in the state* $|0\rangle$. In a circuit diagram, such an operation is represented by a CNOT gate with an *empty* circle on the control qubit, as shown below on the left-hand side. Verify that:



(b) Show that one can swap the role of the control and the target qubit gate by applying four Hadamard gates $\hat{H}$:

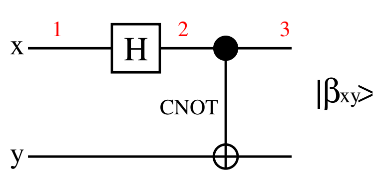


(2) **Bell states**
In the lecture, we defined the Bell states as

$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \ , \qquad |\beta_{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \ ,$$

$$|\beta_{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \ , \text{ and} \qquad |\beta_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \ .$$

(a) Verify that the circuit

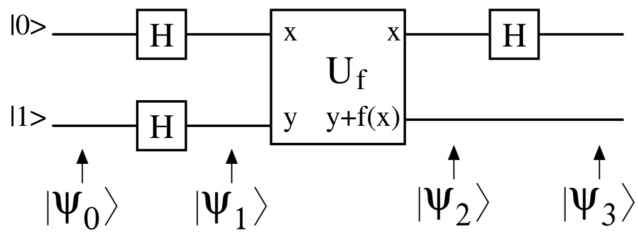


generates the Bell state $|\beta_{xy}\rangle$ if the input state is $|xy\rangle$ with $x, y \in \{0, 1\}$.

(b) Imagine that one of the qubits of a Bell state $|\beta_{xy}\rangle$ is sent to Alice and the other qubit of the same Bell state is sent to Bob (*i.e.*, Alice and Bob share a Bell state). Alice and Bob both apply a Hadamard gate $\hat{H}$ to their qubit. Show that two of the Bell states are interchanged, and two of the Bell states remain unchanged by this transformation (up to a global phase).

(3) **Deutsch algorithm**

Consider a binary classical function $f(x) : \{0,1\} \to \{0,1\}$. The Deutsch algorighm allows us to decide if $f(x)$ is balanced [*i.e.*, $f(0) \neq f(1)$] or constant [*i.e.*, $f(0) = f(1)$]. A quantum circuit implementing the Deutsch algorithm is:



$\hat{U}_f$ is a two-qubit gate that implements the transformation $\hat{U}_f|x,y\rangle = |x, y \oplus f(x)\rangle$, where the symbol $\oplus$ denotes addition modulo 2.

(a) Write down the state $|\psi_1\rangle$ obtained when the first two Hadamard gates $\hat{H}$ have acted on the input state $|\psi_0\rangle = |01\rangle$.

(b) Show that

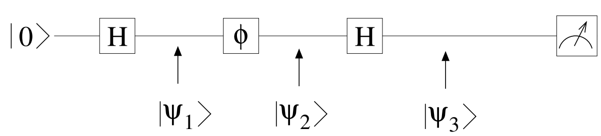
$$|\psi_2\rangle = \begin{cases} \pm\frac{|0\rangle+|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle-|1\rangle}{\sqrt{2}} & \text{if } f(0) = f(1)\ , \\ \pm\frac{|0\rangle-|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle-|1\rangle}{\sqrt{2}} & \text{if } f(0) \neq f(1)\ . \end{cases}$$

(c) Show that the final state is

$$|\psi_3\rangle = \pm|f(0) \oplus f(1)\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}\ .$$

(4) **Interferometers**

The circuit below shows a single-qubit model of an interferometer:



The gate $\hat{\phi}$ maps $|0\rangle \to |0\rangle$ and $|1\rangle \to e^{i\phi}|1\rangle$.

(a) Calculate the states $|\psi_1\rangle$, $|\psi_2\rangle$, and $|\psi_3\rangle$.

(b) What is the probability of measuring zero in the end?