

Theory of Superconductivity, Frühjahrsemester 2026

Blatt 5

Abgabe: 09.04.26, 12:00H (auf adam oder Treppenhause 4. Stock)

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(1) **Derivation of the Ginzburg-Landau equations** (10 Punkte)

The Ginzburg-Landau free energy functional of a superconductor (filling a volume Ω with surface $\partial\Omega$) has the form

$$F = F_n + \int_{\Omega} d^3r \left(\alpha |\psi|^2 + \frac{\beta}{2} |\psi|^4 + \frac{1}{2m^*} \left| \left(\frac{\hbar}{i} \nabla - e^* \mathbf{A} \right) \psi \right|^2 + \frac{\mathbf{h}^2}{2\mu_0} \right), \quad (1)$$

where $\psi(\mathbf{r})$ is the order parameter, $\mathbf{A}(\mathbf{r})$ is the vector potential with $\mathbf{h} = \nabla \times \mathbf{A}$, and F_n is the free energy of the normal state that is a constant. To find the equilibrium state of the system, we look for the minimum of F under variations of ψ and \mathbf{A} . Instead of using the phase and amplitude of ψ as independent parameters, we will consider $F = F[\psi, \psi^*, \mathbf{A}]$ to be a functional of ψ , ψ^* , and \mathbf{A} where ψ and ψ^* are independent variables.

- Find the variation δF of F when ψ^* changes by an amount $\delta\psi^*$.
- Transform the expression obtained in (a) by using the general formula $\nabla \cdot (g\mathbf{f}) = g\nabla \cdot \mathbf{f} + \mathbf{f} \cdot \nabla g$ to eliminate the term containing $\nabla \delta\psi^*$.
- Assuming that there is no current passing through the surface of the superconductor, that is $(\frac{\hbar}{i}\nabla - e^*\mathbf{A})\psi$ has no component perpendicular to $\partial\Omega$, derive the first Ginzburg-Landau equation:

$$\alpha\psi + \beta |\psi|^2 \psi + \frac{1}{2m^*} \left(\frac{\hbar}{i} \nabla - e^* \mathbf{A} \right)^2 \psi = 0.$$

- Find the variation δF of F when \mathbf{A} changes by an amount $\delta\mathbf{A}$.
- Transform the expression obtained in (d) by using the general property $\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$, in order to introduce in δF a term of the form $\nabla \cdot (\mathbf{h} \times \delta\mathbf{A})$.
- Show that the term proportional to $\nabla \cdot (\mathbf{h} \times \delta\mathbf{A})$ obtained in (e) cancels with the volume integration and use the Maxwell equation $\mu_0 \mathbf{J}_s = \nabla \times \mathbf{h}$ to derive the second Ginzburg-Landau equation:

$$\mathbf{J}_s = \frac{e^* \hbar}{2m^* i} (\psi^* \nabla \psi - \psi \nabla \psi^*) - \frac{(e^*)^2}{m^*} \psi^* \psi \mathbf{A}.$$

- Use the notation $\psi = |\psi|e^{i\varphi}$ to give another expression for \mathbf{J}_s . Identify the superfluid velocity \mathbf{v}_s in this expression.