Departement Physik, Universität Basel
Prof. C. Bruder (Zimmer 4.2, Tel.: 207 36 92, Christoph.Bruder@unibas.ch)

Theory of Superconductivity, Frühjahrsemester 2026

Blatt 3

Abgabe: 19.03.26, 12:00H (auf adam oder Treppenhaus 4. Stock)
Tutor: Tobias Nadolny, Zi.: 4.48

(1) **Specific heat of a superconductor**

The elementary excitations with momentum $\mathbf{k}$ of a superconductor have excitation energy $E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + \Delta^2(T)}$, where $\xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu$ and $\Delta(T)$ is the temperature-dependent gap function. They are non-interacting fermions, i.e., their occupation number is given by the Fermi function $f_{\mathbf{k}} = (e^{\beta E_{\mathbf{k}}} + 1)^{-1}$, where $\beta = (k_B T)^{-1}$.

(a) The entropy of a system of non-interacting fermions is given by

$$S = -2k_B \sum_{\mathbf{k}} \left[(1 - f_{\mathbf{k}}) \ln(1 - f_{\mathbf{k}}) + f_{\mathbf{k}} \ln f_{\mathbf{k}}\right] .$$

Show that the specific heat $c = \frac{1}{\text{Vol.}} T \frac{dS}{dT}$ can be written as

$$c = \frac{2}{\text{Vol.}} \beta k_B \sum_{\mathbf{k}} \left(-\frac{\partial f_{\mathbf{k}}}{\partial E_{\mathbf{k}}}\right) \left(E_{\mathbf{k}}^2 + \frac{1}{2}\beta \frac{d\Delta^2}{d\beta}\right) . \qquad (1)$$

Hint: $T\frac{d}{dT} = -\beta \frac{d}{d\beta}$.

Equation (1) is a bit tedious to obtain. If you do not manage, use the expression (1) and proceed with (b) and (c).

(b) Prove that in the limit $\Delta(T) \to 0$, i.e., for a normal metal, we obtain the well-known formula

$$c_{\text{n}} = \frac{2\pi^2}{3} N(0) k_B^2 T$$

where $N(0)$ is the density of states (per spin) of the free Fermi gas at the Fermi energy.

(c) The temperature dependence of Δ close to T_C is approximately given by

$$\Delta(T) = 3.06 k_B \sqrt{T_C} \sqrt{T_C - T} \quad \text{for} \quad T \lesssim T_C ,$$

and (of course) $\Delta(T) = 0$ for $T > T_C$.

Calculate the jump $\frac{c - c_{\text{n}}}{c_{\text{n}}}$ in the specific heat at $T = T_C$.

(d) What is the temperature dependence of c for low temperatures, $k_B T/\Delta(0) \to 0$?

(2) **Superconducting density of states**
The excitation energy of the quasi-particle is given by

$$E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + \Delta^2}\,,$$

where $\xi_{\mathbf{k}}$ is the independent-particle kinetic energy relative to the Fermi energy.
The number of excitations in a superconductor in the energy interval $[E, E + dE]$ is $N_s(E)dE$, here, $N_s(E)$ is the density of states of the superconductor. The corresponding number in the normal state is $N_n(\xi)d\xi \approx N_n(0)d\xi$, here, $N_n(\xi)$ is the density of states of the normal metal. Calculate $N_s(E)/N_n(0)$ by equating the two expressions.
Compare the superconducting density of states with the normal one and discuss the origin of the divergence of $N_s(E)$ at $E = \Delta$.

(3) **Temperature dependence of Δ**

The temperature dependence of $\Delta(T)$ is determined by the gap equation

$$\Delta_{\mathbf{k}} = -\sum_{\mathbf{l}} V_{\mathbf{kl}} \frac{\Delta_{\mathbf{l}}}{2E_{\mathbf{l}}} \tanh(\frac{\beta E_{\mathbf{l}}}{2}) \tag{2}$$

where $E_{\mathbf{l}} = \sqrt{\xi_{\mathbf{l}}^2 + \Delta_{\mathbf{l}}^2}$ and $\beta = (k_B T)^{-1}$.

(a) Assuming that $V_{\mathbf{kl}} = -V$ if $|\xi_{\mathbf{k}}|$, $|\xi_{\mathbf{l}}| < \hbar\omega_c$ and 0 otherwise, show that the gap equation can be written as

$$1 = VN(0) \int_0^{\hbar\omega_c} \mathrm{d}\xi \frac{1}{E} \tanh(\frac{\beta}{2}E) \tag{3}$$

where $E = \sqrt{\xi^2 + \Delta^2}$. Use (3) at $T = T_C$ to eliminate $VN(0)$ in favor of T_C and solve the resulting equation to obtain $\Delta(T)$ numerically in the temperature interval $[0, 2T_C]$.

(b) Study the behavior of $\Delta(T)$ for $T \to T_C$ and for $T \to 0$, either analytically or numerically.

(4) **Ground-state energy**
Define $\langle E \rangle = \langle \psi_{\mathrm{BCS}} | H - \mu\hat{N} | \psi_{\mathrm{BCS}} \rangle$. Follow the steps in Tinkham 3.4.2 to evaluate the difference between the ground-state energy of the BCS state and the free Fermi gas.
Result:

$$\langle E \rangle_s - \langle E \rangle_n = -\frac{1}{2} N(0) \Delta^2$$