Departement Physik, Universität Basel
Prof. C. Bruder (Zimmer 4.2, Tel.: 061 207 36 92, Christoph.Bruder@unibas.ch)

Classical and Quantum Nonlinear Dynamics
Frühjahrsemester 2024

Blatt 8

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Tutor: Parvinder Solanki Zi. 448; parvinder.parvinder@unibas.ch

(1) **Kuramoto's proof that drifters do not contribute** (2 Punkte)
Kuramoto noted that the stationary distribution of drifting oscillators satisfies the following symmetry:

$$\rho(\theta, \omega) = \rho(\theta + \pi, -\omega)$$

for $|\omega| > Kr$. Use that symmetry and the assumption that the density $g(\omega)$ of natural frequencies is even to prove that

$$\langle e^{i\theta} \rangle_{\text{drift}} = 0$$

in steady state.

(2) **Square-root growth of the Kuramoto order parameter r near K_c** (3 Punkte)
The goal of this exercise is to show that $r_\infty(K)$ grows like $(K - K_c)^{1/2}$ for K just above K_c if the unimodal density g has a quadratic maximum.

(a) Expand $g(\omega)$ in a Taylor series for small ω. Why is it appropriate to assume that ω is small?

(b) Substitute $\omega = Kr \sin\theta$ into your Taylor series formula. Explain what this substitution means and why it is valid.

(c) Assuming that r is small when K is near K_c, show that the self-consistency equation implies

$$r \approx b\sqrt{K - K_c}$$

and find the prefactor b explicitly in terms of K_c and $g''(0)$.

(3) **Locking threshold for parabolic density of natural frequencies** (5 Punkte)
Consider the following density for the natural frequencies in the Kuramoto model:

$$g(\omega) = \frac{3}{4}(1 - \omega^2)$$

for $-1 \leq \omega \leq 1$, and $g(\omega) = 0$ otherwise. In cases like this where the density does not have infinite tails, the Kuramoto model undergoes *two* transitions: a transition from incoherence to partial locking at $K = K_c$ followed by a transition from partial locking to complete locking at $K = K_L \geq K_c$.

(a) Determine K_c and K_L by choosing approproate values of Kr in the integrand of the self-consistency equation.
Hint for K_L: what is the minimal value of Kr such that all phases will lock?

(b) Show that the order parameter for the partially-locked states is given by

$$r_\infty(K) = \frac{2}{K}\sqrt{1 - \frac{8}{3\pi K}}\,.$$

(c) For $K > K_L$ the system is completely locked. The phases of the locked oscillators no longer span the whole semicircle $|\theta| \leq \pi/2$, but squeeze together more tightly. Find the maximum locked phase $\theta_{\max}$.

(d) (Bonus problem) To compute $r_\infty(K)$ for the completely locked states, we can use the parametric equation method of problem 13.5.5 in Strogatz.
Define $u = Kr$, rewrite $\theta_{\max}$ in terms of u and modify the limits of integration in the self-consistency equation accordingly. Then do the same for $f(u)$. Show that

$$f(u) = \frac{3}{16}\left(\frac{\sqrt{u^2-1}(u^2+2)}{u^2} - (u^2-4)\csc^{-1}(u)\right)\,.$$

Find $r(u)$ and $K(u)$ and use them to plot $r(K)$ for the completely-locked states.

(e) (Bonus problem) As an alternative to (d), simulate the Kuramoto model numerically for the parabolic density and try to estimate K_c and K_L.