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Classical and Quantum Nonlinear Dynamics
Frühjahrsemester 2024

Blatt 7

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(1) **Mutual synchronization of two interacting self-oscillators** (3 Punkte)

(a) Read and understand p. 222 – 225 in Pikovsky *et al.*'s book.

(b) In the following we focus on the case of 1:1 resonance. Integrate numerically Eq. (8.8) for fixed $\epsilon = -1$ and use the result to plot the observed frequency difference between the two oscillators, $\langle \dot{\psi} \rangle$, as a function of detuning ν. Highlight the region at small detuning in which the two oscillators synchronize. What value does $\langle \dot{\psi} \rangle$ approach for large values of ν?

(2) **Two coupled vdP oscillators** (4 Punkte + 2 Bonuspunkte)
Consider the following system of two coupled van der Pol oscillators,

$$\ddot{x}_A + \mu_A(x_A^2 - 1)\dot{x}_A + \omega_A x_A = gF(x_B, \dot{x}_B), \tag{1}$$

$$\ddot{x}_B + \mu_B(x_B^2 - 1)\dot{x}_B + \omega_B x_B = gG(x_A, \dot{x}_A), \tag{2}$$

where the nature of the coupling depends on the linear functions F and G. Solve it numerically for (i) coherent coupling: $F(x_B, \dot{x}_B) = x_B$, $G(x_A, \dot{x}_A) = x_A$ and (ii) dissipative coupling: $F(x_B, \dot{x}_B) = \dot{x}_B$, $G(x_A, \dot{x}_A) = \dot{x}_A$ with the following choices of parameters:

(a) $\omega_A = 1$, $\omega_B = 1.2$, $\mu_A = \mu_B = 1$, and $g = \{0, 0.1, 0.5\}$. Show that the two vdP oscillators get synchronized for $g = 0.5$ and remain unsynchronized for $g = 0.1$.

(b) $\omega_A = 1$, $\omega_B = 1.2$, $\mu_A = 1$, $\mu_B = -1$, and $g = \{0, 0.1, 0.3\}$. Choose initial states such that $|x_\alpha| < 1$ and $|\dot{x}_\alpha| < 1$ where $\alpha = \{A, B\}$. Compare with (a) and discuss your results.

(c) Bonus problem: Produce a plot similar to the one produced in 1 (b) for two coupled vdP oscillators. The detuning enters by setting $\omega_A = 1 + \nu/2$ and $\omega_B = 1 - \nu/2$. The coupling is fixed at $g = 1$, and $\mu_A = \mu_B = 1$. Try both dissipative and coherent coupling.
Hint: The frequency can be estimated as 1 over the period.

(3) **Kuramoto model with $N = 1000$ oscillators** (3 Punkte + 2 Bonuspunkte)
In the lecture we defined the order parameter r of the Kuramoto model as

$$re^{i\psi} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j} \, .$$

(a) Show that the coupling term of the Kuramoto model can be expressed in terms of r and ψ.

(b) Numerically integrate the Kuramoto model, using 1000 oscillators whose natural frequencies are randomly sampled from a Gaussian distribution with mean $\mu = 5$ and standard deviation $\sigma = 1$.
Fix the coupling strength at $K = 1$. Start all the oscillators in phase, and compute the time evolution of the order parameter $r(t)$. What happens in the long run?

(c) Redo the simulation above at $K = 2.5$. Start the system with a low value of r by scattering the initial phases uniformly at random over the interval $0 < \theta < 15\pi/8$. Compute the order parameter $r(t)$ until it settles down.