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Classical and Quantum Nonlinear Dynamics
Frühjahrsemester 2024

Blatt 2

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(1) **Distinguishing various types of bifurcations** (2 Punkte)
In each case, find the values of r at which bifurcations occur, and classify those as saddle-node, transcritical, supercritical pitchfork, or subcritical pitchfork. Finally, sketch the bifurcation diagrams of fixed points x^* vs. r.

(a) $\dot{x} = rx - x/(1+x)$
(b) $\dot{x} = rx - x/(1+x^2)$
(c) $\dot{x} = x + \tanh(rx)$
(d) $\dot{x} = rx + x^3/(1+x^2)$

(2) **Flows on the circle** (6 Punkte)
Read and understand Sections 4.1 - 4.3 and 4.6 in Strogatz.

(a) Consider a Josephson junction in the overdamped limit $\beta = 0$ described by

$$\phi' := \frac{\hbar}{2eRI_c}\dot{\phi} = \frac{I}{I_c} - \sin\phi \qquad (1)$$

(Eq. (4.6.7) in Strogatz; $\phi' = \mathrm{d}\phi/\mathrm{d}\tau$ where $\tau = 2eRI_c t/\hbar$ is a dimensionless time.) Sketch the supercurrent $I_c \sin\phi(t)$ as a function of t, assuming first that I/I_c is slightly greater than 1, and then assuming that $I/I_c \gg 1$.

(b) Sketch the instantaneous voltage $V(t) = (\hbar/(2e))\dot{\phi}(t)$ for the two cases considered in (a)

(c) Check your qualitative conclusions in (a) and (b) by integrating (1) numerically, and plotting the graphs of $I_c \sin\phi(t)$ and $V(t)$. Calculate the time average of $\langle V\rangle$ of $V(t)$ and compare with the exact result $\langle V\rangle = I_c R\sqrt{(I/I_c)^2 - 1}$ for $I > I_c$.

(3) **Phase portraits** (2 Punkte)
Read Section 4.1 in Strogatz. For each of the following questions, sketch the phase portrait as a function of the control parameter μ. Classify the bifurcations that occur as μ varies, and find all the bifurcation values of μ.
(a) $\dot{\theta} = \mu\sin\theta - \sin 2\theta$ (b) $\dot{\theta} = \sin\theta/(\mu + \cos\theta)$ (c) $\dot{\theta} = \mu + \cos\theta + \cos 2\theta$
(d) $\dot{\theta} = \sin 2\theta/(1 + \mu\sin\theta)$

(4) **Critical slowing down** (3 Bonuspunkte)

In statistical mechanics, the phenomenon of "critical slowing down" is a signature of a second-order phase transition. At the transition, the system relaxes to equilibrium much more slowly than usual. A mathematical version of the effect can be studied using the normal form of the supercritical pitchfork bifurcation $\dot{x} = rx - x^3$.

(a) For $r < 0$, the origin is the only fixed point, and it is stable. Determine the typical time of the decay towards the origin.

(b) At $r = 0$, the origin is still stable, but much more weakly so. Obtain the analytical solution to $\dot{x} = -x^3$ for an arbitrary initial condition. Show that $x(t) \to 0$ as $t \to \infty$, but that the decay is not exponential.

(c) To get some intuition about the slowness of the decay, make a numerically accurate plot of the solution for the initial condition $x(0) = 1$, for $0 \leq t \leq 10$, and compare the decay for $r = -1$ and $r = 0$.