Departement Physik, Universität Basel
Prof. C. Bruder (Zimmer 4.2, Tel.: 207 36 92, Christoph.Bruder@unibas.ch)

Theory of Superconductivity, Frühjahrsemester 2023

Blatt 6

Abgabe: 20.4.23, 12:00H (Treppenhaus 4. Stock)
Tutor: Tobias Nadolny, Zi.: 4.48

(1) **Derivation of the Ginzburg-Landau equations** (10 Punkte)

The Ginzburg-Landau free energy functional of a superconductor (filling a volume Ω with surface $\partial\Omega$) has the form

$$F = F_n + \int_\Omega d^3r \, \left(\alpha \left|\psi\right|^2 + \frac{\beta}{2}\left|\psi\right|^4 + \frac{1}{2m^*}\left|(\frac{\hbar}{i}\nabla - e^*\mathbf{A})\psi\right|^2 + \frac{\mathbf{h}^2}{2\mu_0}\right) \, , \qquad (1)$$

where $\psi(\mathbf{r})$ is the order parameter, $\mathbf{A}(\mathbf{r})$ is the vector potential with $\mathbf{h} = \nabla \times \mathbf{A}$, and F_n is the free energy of the normal state that is a constant. To find the equilibrium state of the system, we look for the minimum of F under variations of ψ and $\mathbf{A}$. Instead of using the phase and amplitude of ψ as independent parameters, we will consider $F = F[\psi, \psi^*, \mathbf{A}]$ to be a functional of ψ, ψ^*, and $\mathbf{A}$ where ψ and ψ^* are independent variables.

(a) Find the variation δF of F when ψ^* changes by an amount $\delta\psi^*$.

(b) Transform the expression obtained in (a) by using the general formula $\nabla \cdot (g\mathbf{f}) = g\nabla \cdot \mathbf{f} + \mathbf{f} \cdot \nabla g$ to eliminate the term containing $\nabla\delta\psi^*$.

(c) Assuming that there is no current passing through the surface of the superconductor, that is $(\frac{\hbar}{i}\nabla - e^*\mathbf{A})\psi$ has no component perpendicular to $\partial\Omega$, derive the first Ginzburg-Landau equation:

$$\alpha\psi + \beta\left|\psi\right|^2\psi + \frac{1}{2m^*}(\frac{\hbar}{i}\nabla - e^*\mathbf{A})^2\psi = 0 \, .$$

(d) Find the variation δF of F when $\mathbf{A}$ changes by an amount $\delta\mathbf{A}$.

(e) Transform the expression obtained in (d) by using the general property $\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b})$, in order to introduce in δF a term of the form $\nabla\cdot(\mathbf{h}\times\delta\mathbf{A})$.

(f) Show that the term proportional to $\nabla\cdot(\mathbf{h}\times\delta\mathbf{A})$ obtained in (e) cancels with the volume integration and use the Maxwell equation $\mu_0\mathbf{J}_s = \nabla\times\mathbf{h}$ to derive the second Ginzburg-Landau equation:

$$\mathbf{J}_s = \frac{e^*\hbar}{2m^* i}(\psi^*\nabla\psi - \psi\nabla\psi^*) - \frac{(e^*)^2}{m^*}\psi^*\psi\mathbf{A} \, .$$

(g) Use the notation $\psi = |\psi|e^{i\varphi}$ to give another expression for $\mathbf{J}_s$. Identify the superfluid velocity $\mathbf{v}_s$ in this expression.