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Theory of Superconductivity, Frühjahrsemester 2023

Blatt 4

Abgabe: 30.3.23, 12:00H (Treppenhaus 4. Stock)
Tutor: Tobias Kehrer, Zi.: 4.48

(1) **Specific heat of a superconductor** (7 Punkte)

The elementary excitations with momentum $\mathbf{k}$ of a superconductor have excitation energy $E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + \Delta^2(T)}$, where $\xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu$ and $\Delta(T)$ is the temperature-dependent gap function. They are non-interacting fermions, i.e., their occupation number is given by the Fermi function $f_{\mathbf{k}} = (e^{\beta E_{\mathbf{k}}} + 1)^{-1}$, where $\beta = (k_B T)^{-1}$.

(a) The entropy of a system of non-interacting fermions is given by

$$S = -2k_B \sum_{\mathbf{k}} [(1 - f_{\mathbf{k}}) \ln(1 - f_{\mathbf{k}}) + f_{\mathbf{k}} \ln f_{\mathbf{k}}] \,.$$

Show that the specific heat $c = \frac{1}{\text{Vol.}} T \frac{dS}{dT}$ can be written as

$$c = \frac{2}{\text{Vol.}} \beta k_B \sum_{\mathbf{k}} \left(-\frac{\partial f_{\mathbf{k}}}{\partial E_{\mathbf{k}}} \right) \left(E_{\mathbf{k}}^2 + \frac{1}{2} \beta \frac{d\Delta^2}{d\beta} \right) \,. \tag{1}$$

Hint: $T \frac{d}{dT} = -\beta \frac{d}{d\beta}$.

Equation (1) is a bit tedious to obtain. If you do not manage, use the expression (1) and proceed with (b) and (c).

(b) Prove that in the limit $\Delta(T) \to 0$, i.e., for a normal metal, we obtain the well-known formula

$$c_{\text{n}} = \frac{2\pi^2}{3} N_{\text{n}}(0) k_B^2 T \,.$$

(c) The temperature dependence of Δ close to T_C is approximately given by

$$\Delta(T) = 3.06 k_B \sqrt{T_C} \sqrt{T_C - T} \quad \text{for} \quad T \lesssim T_C \,,$$

and (of course) $\Delta(T) = 0$ for $T > T_C$.

Calculate the jump $\frac{c - c_{\text{n}}}{c_{\text{n}}}$ in the specific heat at $T = T_C$.

(d) What is the temperature dependence of c for low temperatures, $k_B T / \Delta(0) \to 0$?

(2) **Semiconductor analogy** (3 Punkte)

To understand quasiparticle tunneling intuitively it is useful to describe superconductors via the so-called *semiconductor analogy* that treats the ground state like a filled "valence band" of negative-energy solutions. To justify this picture, we introduce (yet) another set of positive-energy (negative-energy) quasiparticle operators $\alpha^\dagger_{\mathbf{k}\sigma+}$ ($\alpha^\dagger_{\mathbf{k}\sigma-}$) where $\mathbf{k}$ is assumed to lie only in one half of $\mathbf{k}$-space (such that the set of all $\mathbf{k}$ and all $-\mathbf{k}$ covers all of $\mathbf{k}$-space):

$$\begin{aligned} \alpha^\dagger_{\mathbf{k}\sigma+} &= \gamma^\dagger_{\mathbf{k}\sigma} \\ \alpha^\dagger_{\mathbf{k}\sigma-} &= \mathrm{sgn}(\sigma)\gamma_{-\mathbf{k}-\sigma} \, . \end{aligned}$$

Here, the γ's are the Bogoliubov operators that we introduced in the lecture.

(a) Rewrite the diagonalized BCS (Bogoliubov) Hamiltonian

$$H_M = \sum_{\text{all } \mathbf{k}} E_{\mathbf{k}} \left(\gamma^\dagger_{\mathbf{k}\uparrow}\gamma_{\mathbf{k}\uparrow} + \gamma^\dagger_{-\mathbf{k}\downarrow}\gamma_{-\mathbf{k}\downarrow} \right) + \text{const.}$$

using these new operators and show that the excitation spectrum can be interpreted in terms of an empty band of positive-energy excitations and a filled band of negative-energy excitations.

(b) Show that the BCS ground state can be regarded as a filled sea of negative-energy quasiparticle states and an empty sea of positive-energy quasiparticle states.

(3) **Transition probabilities and coherence effects** (5 Extra-Punkte)

Read and understand the Section *Calculation of Transition Probabilities* in de Gennes' book (p. 131 ff.) and explain in your own words why the (ultra-)sound attenuation in a superconductor is lowered below T_C whereas the NMR relaxation rate is increased.