Departement Physik, Universität Basel
Prof. C. Bruder (Zimmer 4.2, Tel.: 207 36 92, Christoph.Bruder@unibas.ch)

Theory of Superconductivity, Frühjahrsemester 2023

Blatt 2

Abgabe: 16.3.23, 12:00H (Treppenhaus 4. Stock)
Tutor: Tobias Nadolny, Zi.: 4.48

To obtain **4 credit points** and the grade "4" you will have to reach 50% of the points in the homework problems. The grade can be improved by participating in the oral exam.

---

(1) **Quasi-particle excitations in superconductors** (5 Punkte)

Define operators $\gamma^\dagger_{\mathbf{k}\uparrow}$ and $\gamma^\dagger_{\mathbf{k}\downarrow}$ by

$$\gamma^\dagger_{\mathbf{k}\uparrow} = u^*_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} - v^*_{\mathbf{k}} c_{-\mathbf{k}\downarrow}$$
$$\gamma^\dagger_{-\mathbf{k}\downarrow} = u^*_{\mathbf{k}} c^\dagger_{-\mathbf{k}\downarrow} + v^*_{\mathbf{k}} c_{\mathbf{k}\uparrow} \,;$$

$u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ are complex numbers satisfying $|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2 = 1$ for each momentum $\mathbf{k}$, and $c^\dagger$, c are the (standard) electron creation and annihilation operators.

(a) Prove that the superconducting ground state $|\psi_{\mathrm{BCS}}\rangle = \prod_{\mathbf{k}}(u_{\mathbf{k}} + v_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow})\,|0\rangle$ is the vacuum state of the γ operators, that is,

$$\gamma_{\mathbf{k}\uparrow}\,|\psi_{\mathrm{BCS}}\rangle = \gamma_{\mathbf{k}\downarrow}\,|\psi_{\mathrm{BCS}}\rangle = 0\,.$$

(b) Obtain explicit expressions of the states created by the $\gamma^\dagger$ operators, $\gamma^\dagger_{\mathbf{k}\uparrow}\,|\psi_{\mathrm{BCS}}\rangle$ and $\gamma^\dagger_{\mathbf{k}\downarrow}\,|\psi_{\mathrm{BCS}}\rangle$, in terms of the electron creation operators $c^\dagger_{\mathbf{k}\uparrow}$ and $c^\dagger_{\mathbf{k}\downarrow}$.

We will see that the states created by $\gamma^\dagger_{\mathbf{k}\uparrow}$, $\gamma^\dagger_{\mathbf{k}\downarrow}$ are the quasi-particle excitations of wave vector $\mathbf{k}$ and spin $\uparrow$ and $\downarrow$ above the superconducting ground state.

(2) **Average and fluctuations of electron number in the BCS state** (5 Punkte)

(a) Obtain the average electron number $\overline{N} = \langle\psi_{\mathrm{BCS}}|N|\psi_{\mathrm{BCS}}\rangle$ in the BCS ground state in terms of $v_{\mathbf{k}}$ or $u_{\mathbf{k}}$, where the total electron number operator N has the following second-quantized form:

$$N = \sum_{\mathbf{k}} \left(c^\dagger_{\mathbf{k}\uparrow} c_{\mathbf{k}\uparrow} + c^\dagger_{\mathbf{k}\downarrow} c_{\mathbf{k}\downarrow} \right)\,.$$

Interpret the result in view of the physical meaning of $v_{\mathbf{k}}$ or $u_{\mathbf{k}}$.
Hint: one way (but not the only way) is to rewrite the electron operators in terms of the γ-operators and then use the result of problem 1(a).

(b) Obtain the fluctuation of the electron number $(\delta N)^2 = \langle\psi_{\mathrm{BCS}}|(N-\overline{N})^2|\psi_{\mathrm{BCS}}\rangle$ in a similar way as done in (a). How does $\delta N/\overline{N}$ behave in the thermodynamic limit $\overline{N}\to\infty$?

(c) (independent of (a) and (b)) Show that

$$|\psi_{\mathrm{N}}\rangle = \left(\sum_{\mathbf{k}} g_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow}\right)^{\frac{N}{2}} |0\rangle$$

can be obtained by projecting $|\psi_{\mathrm{BCS}}\rangle$ on the subspace of states with particle number N. How is $g_{\mathbf{k}}$ related to $u_{\mathbf{k}}$, $v_{\mathbf{k}}$?