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Theory of Superconductivity, Frühjahrsemester 2023

Blatt 1

Abgabe: 2.3.23, 12:00H (Treppenhaus 4. Stock)
Tutor: Tobias Nadolny, Zi.: 4.48

(1) **Meissner effect** (5 Punkte)

The current response of superconductors (at least close to the transition temperature) is described by the London equation,

$$\mathbf{j}(\mathbf{r}) = -\frac{1}{\mu_0 \lambda^2} \mathbf{A}(\mathbf{r}) \, ,$$

where $\mathbf{A}(\mathbf{r})$ is the vector potential in the Coulomb gauge ($\boldsymbol{\nabla} \cdot \mathbf{A} = 0$) and λ is a material-specific length. (Further details are described in Tinkham, Sec. 1.2, but they are not relevant for solving the exercise).

Consider a superconducting semi-infinite space ($x > 0$) exposed to an external homogeneous magnetic field $\mathbf{B}_0 = B_0 \mathbf{e}_z$ in the z-direction. Calculate and plot the magnetic field and the current density in the superconductor.

(2) **Toy model of Bogoliubov transformation** (5 Punkte)

The effective Hamiltonian

$$H = \epsilon_a a^\dagger a + \epsilon_b b^\dagger b - \Delta b a - \Delta^* a^\dagger b^\dagger \, ,$$

contains fermions in two kinds of states a und b (i.e., $\{a, a^\dagger\} = 1$, $\{a, b\} = 0$ etc., here, $\{,\}$ is the anticommutator). We would like to diagonalize this Hamiltonian, i.e., express it in the form

$$H = E_\alpha \alpha^\dagger \alpha + E_\beta \beta^\dagger \beta + E_0$$

by introducing the so-called quasiparticle operators α, β through the following unitary transformation:

$$a^\dagger = u\alpha^\dagger + v\beta \, , \quad b = -v^* \alpha^\dagger + u^* \beta \, .$$

(u, v are complex numbers, α, β are fermionic operators!)

(a) Show that the coefficients have to fulfill $|u|^2 + |v|^2 = 1$.

(b) Express H through α and β, and determine u and v such that H is diagonalized. Determine the energy spectrum of the new quasiparticles, that is, find the expressions for E_α and E_β for the special case $\epsilon_a = \epsilon_b = \epsilon$, and u, v, Δ are real.

(c) Discuss the meaning of E_α, E_β, and E_0.

(3) **Landau diamagnetism** (5 Bonuspunkte)

To appreciate why the Meissner effect is special we will calculate the orbital magnetic susceptibility of a spinless non-interacting electron gas (particle number N, volume V) at $T = 0$. Assume that an external magnetic field $\mathbf{H}_0(\mathbf{r})$ is produced by a current density $\mathbf{j}_0(\mathbf{r})$. This field will induce a current density $\langle \mathbf{j}(\mathbf{r})\rangle = \langle \frac{e}{2}\sum_i[\mathbf{v}_i\delta(\mathbf{r}-\mathbf{r}_i) + \delta(\mathbf{r}-\mathbf{r}_i)\mathbf{v}_i]\rangle$ in the electron gas, and the resulting total field $\mu_0\mathbf{H} = \boldsymbol{\nabla}\times\mathbf{A}$ obeys the Maxwell equation $\boldsymbol{\nabla}\times\mathbf{H} = \langle \mathbf{j}_0 + \mathbf{j}\rangle$, where $\langle ...\rangle$ is the ground-state expectation value and $\langle \mathbf{j}\rangle$ is connected to the magnetization via $\langle \mathbf{j}(\mathbf{r})\rangle = \boldsymbol{\nabla}\times\mathbf{M}(\mathbf{r})$.

(a) Show by a Fourier transform that the magnetic susceptibility χ defined by $\mathbf{M}(\mathbf{q}) = \chi(\mathbf{q})\mathbf{H}_0(\mathbf{q})$ can be written as

$$\chi(\mathbf{q}) = \frac{\mu_0\langle \mathbf{n}\cdot\mathbf{j}(\mathbf{q})\rangle}{q^2 A(\mathbf{q}) - \mu_0\langle \mathbf{n}\cdot\mathbf{j}(\mathbf{q})\rangle}\,, \tag{1}$$

where $\mathbf{n} \parallel \mathbf{A}$ is a unit vector. Here and in the following we will assume that χ is a scalar, i.e. $\mathbf{M} \parallel \mathbf{H}_0$, and that $\mathbf{A}$ is given in the Coulomb gauge $\mathbf{q}\cdot\mathbf{A}(\mathbf{q}) = 0$.

(b) Show that $\langle \mathbf{j}(\mathbf{q})\rangle$ can be written as $\langle \mathbf{j}(\mathbf{q})\rangle = \frac{e}{m}\langle \mathbf{p}_\mathbf{q}\rangle - \frac{e^2}{m}\frac{N}{V}\mathbf{A}(\mathbf{q})$, where $\mathbf{p}_\mathbf{q}$ is the Fourier-transformed momentum density operator.

(c) To find $\chi(\mathbf{q})$ we will calculate $\langle \mathbf{j}(\mathbf{q})\rangle$ using first-order perturbation theory in $\mathbf{A}$. To first order in $\mathbf{A}$, $\mathcal{H} = \sum_i \frac{1}{2m}[\mathbf{p}_i - e\mathbf{A}(\mathbf{r}_i)]^2$ can be written as $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$ where $\mathcal{H}_1 = -\frac{e}{m}\sum_\mathbf{k}\mathbf{A}(\mathbf{k})\cdot\mathbf{p}_{-\mathbf{k}}$. Consider $\mathcal{H}_1$ as perturbation and use the first-order perturbed groundstate to express $\langle \mathbf{p}_\mathbf{q}\cdot\mathbf{n}\rangle$ as

$$\langle \mathbf{p}_\mathbf{q}\cdot\mathbf{n}\rangle = \frac{2e}{m}\sum_{j\neq 0}\frac{|\langle j|\mathbf{p}_\mathbf{q}\cdot\mathbf{n}|0\rangle|^2}{E_j - E_0}A(\mathbf{q})\,. \tag{2}$$

Here, $|j\rangle$ are the eigenstates of the unperturbed system. Use the second-quantized expression $\mathbf{p}_\mathbf{q}\cdot\mathbf{n} = \frac{1}{V}\sum_\mathbf{k}\hbar\mathbf{k}\cdot\mathbf{n}c^\dagger_\mathbf{k}c_{\mathbf{k}+\mathbf{q}}$ (or find another argument) to calculate the matrix elements and rewrite Eq. (2) as

$$\langle \mathbf{p}_\mathbf{q}\cdot\mathbf{n}\rangle = \frac{2e}{V}\sum_{\substack{|\mathbf{k}|<k_F\\ |\mathbf{k}+\mathbf{q}|>k_F}}\frac{(\mathbf{k}\cdot\mathbf{n})^2}{\mathbf{k}\cdot\mathbf{q}+q^2/2}A(\mathbf{q}) = \frac{2e}{V}\sum_{|\mathbf{k}|<k_F}\frac{(\mathbf{k}\cdot\mathbf{n})^2}{\mathbf{k}\cdot\mathbf{q}+q^2/2}A(\mathbf{q})\,.$$

Plug this result into Eq. (1), convert the sum to an integral and calculate $\chi(q)$. Result:

$$\chi(q) = \chi_L\frac{3}{8\xi^2}\left[1+\xi^2 - \frac{1}{2\xi}(1-\xi^2)^2\ln\left|\frac{1+\xi}{1-\xi}\right|\right]$$

where $\xi = \frac{q}{2k_F}$ and

$$\chi_L = -\mu_0\frac{e^2}{24\pi^2}\frac{k_F}{m} \qquad \text{(spinless case, i.e., degeneracy 1)}\,.$$

(d) Plot $\chi(q)$ as a function of q. Express $\chi(0)$ "nicely" (hint: the Bohr radius a_0 and the fine structure constant α are helpful) and estimate its value in a normal metal. Compare with the case of a superconductor, $\chi(0) = -1$, and show that this can be obtained from the normal-metal result by setting $\langle \mathbf{p}_\mathbf{q}\cdot\mathbf{n}\rangle = 0$.