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Theory of Superconductivity, Früjahrsemester 2022

Blatt 10

Abgabe: 02.06.22, 12:00H (Treppenhaus 4. Stock)
Tutor: Ryan Tan Zi.: 4.10

(1) **Cooper-pair box** (10 Punkte)

Consider the Cooper pair box Hamiltonian

$$\hat{H}_{\text{CPB}} = 4E_C(\hat{N} - n_g)^2 - E_J \cos(\hat{\varphi})$$

where $n_g = -q/(2e) = -C_g V_g/(2e)$.

(a) Show that $[\hat{N}, \hat{\varphi}]_- = i$ leads to $[\hat{N}, e^{i\hat{\varphi}}]_- = -e^{i\hat{\varphi}}$. Use this commutator to express $\cos(\hat{\varphi})$ and $\hat{H}_{\text{CPB}}$ in the eigenbasis $\{|N\rangle\}_{N\in\mathbb{Z}}$ of the charge operator.

(b) For $E_c \gg E_J$, sketch the energy spectrum as a function of n_g.

(c) Consider the point $n_g = N - 1/2$ for an integer $N \in \mathbb{Z}$. Estimate the energy splitting of the two lowest eigenstates.

(d) Solve the Schrödinger equation of the Cooper-pair box

$$E\psi(N) = H\psi(N) = 4E_C\,(N - n_g)^2\,\psi(N) - \frac{E_J}{2}\left[\psi(N-1) + \psi(N+1)\right]$$

numerically. Plot the eigenvalues as a function of n_g for different values of the ratio E_C/E_J and interpret your result. Compare with (b) and (c).

(2) **Josephson junction via scattering matrix analysis (author: Ehud Amitai)** (10 Extra-Punkte)

In this exercise we will analyze the Josephson current using the *scattering matrix*.

Scattering matrix

Transport of electrons can be described via the scattering matrix formalism. The scattering matrix relates the initial state and final state of a system undergoing a scattering process. In the most general way, an incident electron's wavefunction can be expanded in some basis,

$$\Psi^{(\text{in})} = \sum_\alpha a_\alpha \psi_\alpha^{(\text{in})}, \tag{1}$$

while the scattered electron's wave function can be expanded in some other basis

$$\Psi^{(\text{out})} = \sum_\beta b_\beta \psi_\beta^{(\text{out})}. \tag{2}$$

Now, the scattering matrix element is defined as the probability amplitude of finding an electron initially in $\psi_\alpha^{(\text{in})}$ in the final state $\psi_\beta^{(\text{out})}$:

$$S_{\beta,\alpha} = \langle \psi_\beta^{(\text{out})} | \psi_\alpha^{(\text{in})} \rangle \tag{3}$$

For processes in which the particle number is conserved, the scattering matrix is unitary.

<u>Andreev reflection</u>

Let us consider the interface between a normal metal and a superconductor, as can be seen in Fig. 1. We further assume that the metal and superconductor have the same Fermi energy.

- Let us think of an electron with an energy close to the Fermi energy, E_F. Such an electron from the normal metal impinging on to the superconductor is not able to penetrate into the superconductor. This is because the superconductor has an energy gap around the Fermi energy, and therefore there are no available states for the electron to occupy.
- The electron cannot be reflected from the interface as an electron. This is because that to change its momentum from $k \to -k$ an infinite barrier should be present (or barrier height should be $\gg E_F$ as is the standard convention). However, Δ is much smaller than E_F, and therefore such a potential barrier cannot exert enough force to substantially change the electron's momentum.
- So, the electron cannot penetrate into the superconductor, and it also cannot be reflected as an electron. The only resolution out of the mess is for the electron to be reflected as a hole (a hole traveling to the left has the same momentum as an electron traveling to the right). But then what happens to the missing charge of $2e$?? - A Cooper pair is created in the superconductor.

This process, in which an electron impinging on the normal metal-superconductor interface is reflected as a hole is called **Andreev reflection**. A more quantitative analysis (too long to be repeated here) gives for the amplitude for Andreev reflection

$$r_A(E) = e^{i\chi}, \tag{4}$$

with $\chi = -\arccos\left(\frac{E}{\Delta}\right) - \phi$, where E is the energy of the incoming electron as measured from the Fermi energy, and ϕ is the superconducting phase. For a hole impinging on the interface, a similar analysis holds. It is reflected as an electron with probability amplitude

$$r_A(E) = e^{i\tilde{\chi}}, \tag{5}$$

with $\tilde{\chi} = -\arccos\left(\frac{E}{\Delta}\right) + \phi$.

<u>Andreev Bound States</u>

Now we finally consider a Josephson junction, made of two superconductors and a normal metal in between. The two superconductors have the same gap Δ. Let us think of an electron close to E_F in the normal metal. Traveling to the right, it is being Andreev reflected as a hole. This hole travels to the left, again being reflected as an electron. We therefore conclude that this particle is bound in the normal region. Quantum mechanics teaches us that a localized particle has discrete energy levels.

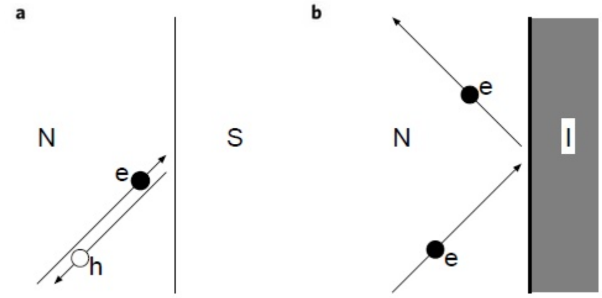

Figure 1: (a). Reflection of an electron in a Normal metal-Superconductor interface. (b). Reflection of an electron in a Normal metal-Insulator interface.

(a) Prove the relation $\mathbf{b} = \mathbf{S}\mathbf{a}$, where $\mathbf{S}$ is the scattering matrix, and $\mathbf{b}$ and $\mathbf{a}$ are the vectors of the coefficients of the scattered wave function (incoming wave function) (see Eq. (1) and Eq. (2)).

We consider a Josephson junction as in Fig. 2. The normal metal is modeled with a scattering matrix $\hat{S}_N$. The scattering matrix of the nanostructure relates the amplitudes of outgoing and incoming states with respect to the nanostructure,

$$\begin{pmatrix} \mathbf{b}_e \\ \mathbf{b}_h \end{pmatrix} = \begin{pmatrix} \hat{S}_N & 0 \\ 0 & \hat{S}_N^* \end{pmatrix} \begin{pmatrix} \mathbf{a}_e \\ \mathbf{a}_h \end{pmatrix} \tag{6}$$

where the two components of the amplitude vectors correspond to the left and right side of the nanostructure, respectively,

$$\mathbf{b}_e = \begin{pmatrix} b_{Le} \\ b_{Re} \end{pmatrix}; \qquad \mathbf{b}_h = \begin{pmatrix} b_{Lh} \\ b_{Rh} \end{pmatrix}, \tag{7}$$

and similarly for the incoming amplitudes $\mathbf{a}_e$, $\mathbf{a}_h$. Also, we define the S_N matrix describing the metal nanostructure to be symmetric (for simplicity),

$$S_N = \begin{pmatrix} r & t \\ -t & r \end{pmatrix}. \tag{8}$$

At the interfaces, as explained, Andreev reflection from the superconductors converts electrons to holes and vice versa, yielding the following relation relation between $\mathbf{a}$ and $\mathbf{b}$:

$$\begin{pmatrix} \mathbf{a}_e \\ \mathbf{a}_h \end{pmatrix} = \begin{pmatrix} 0 & \hat{S}_{eh} \\ \hat{S}_{he} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{b}_e \\ \mathbf{b}_h \end{pmatrix}, \tag{9}$$

with

$$\hat{S}_{eh} = \begin{pmatrix} e^{i\tilde{\chi}_L} & 0 \\ 0 & e^{i\tilde{\chi}_R} \end{pmatrix}; \qquad \hat{S}_{he} = \begin{pmatrix} e^{i\chi_L} & 0 \\ 0 & e^{i\chi_R} \end{pmatrix}. \tag{10}$$

(b) Explain the structure of the matrix in Eq. (9). Why are the diagonal matrix elements zero?

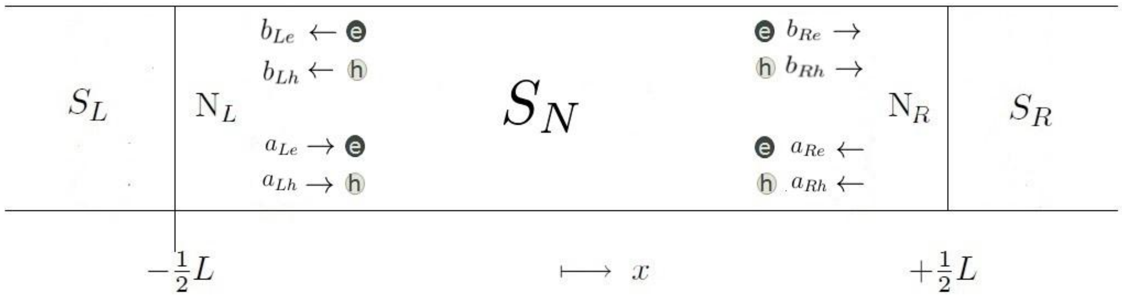

Figure 2:

(c) Use Eq. (6) and Eq. (9) and the folding identity (which holds for square matrices of equal dimensions)

$$\mathrm{Det}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathrm{Det}\left(ad - aca^{-1}b\right), \tag{11}$$

to show that the energy of the bound state is

$$E = \Delta\sqrt{1 - T\sin^2(\phi)/2}, \tag{12}$$

where T is $|t|^2$, and $\phi = \phi_L - \phi_R$ is the phase difference between the two superconductors.

(d) Show that the current in the junction at zero temperature is given by

$$I(\phi) = \frac{2e}{\hbar}\frac{\partial E}{\partial \phi} = \frac{e\Delta}{2\hbar}\frac{T\sin\phi}{\sqrt{1 - T\sin^2(\phi/2)}} \tag{13}$$

[*Hint*: Think of the power IV of the process. Also, use the second Josephson relation $\dot{\phi} = 2eV/\hbar$.]

(e) For perfect transmission $|T| = 1$, what is the current? For a tunneling nanostructure, $|T| \ll 1$, what is the current? Compare the tunneling case with the Ambegaokar-Baratoff relation that we discussed in class.

(3) **Zusatzaufgabe: Shapiro steps** (10 Extra-Punkte)

Assume an ideal Josephson junction that is voltage-biased with

$$V = V_0 + V_1\cos\omega_1 t\,.$$

(a) Calculate the phase difference $\gamma(t)$ across the junction and write down the current $I = I_c\sin\gamma$.

(b) Use the addition theorem and the Fourier expansions of $\sin(z\sin(\theta))$ and $\cos(z\sin(\theta))$ (look them up!) to express the current I as a trigonometric series.

(c) Find the values of V_0 for which the current has a DC component.

(d) Since the integration constant in $\gamma(t)$ is arbitrary, the DC current corresponding to one of the "Shapiro" values in (c) can take different values. Find the width of the corresponding current region. These plateaus are called "Shapiro steps".