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Advanced quantum mechanics and quantum field theory, FS 2021

Blatt 5

Submission: 15.04.2021, 12:00H, **on adam in the appropriate folder.**
One file per submission please; the filename HAS TO contain your name, or the submission will not be corrected!
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(1) **Projection on transverse components** (5 Punkte)

(a) Show that the Lagrangian for electromagnetism can be written as

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu)\,.$$

Hence show that using the transverse projection operator $P^{\mathrm{T}}_{\mu\nu}$, which you have to read up on in Section 13.3 of Lancaster/Blundell, the Lagrangian may be expressed as

$$\mathcal{L} = \frac{1}{2} A^\mu P^{\mathrm{T}}_{\mu\nu} \partial^\sigma \partial_\sigma A^\nu\,.$$

This shows that $\mathcal{L}$ only includes the transverse components of the field.

(b) The transverse δ-function is defined as

$$\delta^{ij}_{\mathrm{tr}}(\mathbf{x}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} e^{i\mathbf{px}} \left(\delta^{ij} - \frac{p^i p^j}{p^2}\right)$$

Show that it is indeed transverse, i.e., that its divergence vanishes.

(c) Show that

$$\delta^{ij}_{\mathrm{tr}}(\mathbf{x}) = \frac{2}{3}\delta^{ij}\delta(\mathbf{x}) - \frac{1}{4\pi x^3}\left(\delta^{ij} - \frac{3x^i x^j}{x^2}\right)\,,$$

i.e., $\delta^{ij}_{\mathrm{tr}}(\mathbf{x})$ does not vanish for $\mathbf{x} \neq 0$. Plot $\delta^{ij}_{\mathrm{tr}}(\mathbf{x})$ along some typical directions.

(d) Show that $\delta^{ij}_{\mathrm{tr}}(\mathbf{x})$ and its cousin $\delta^{ij}_{\mathrm{long}}(\mathbf{x}) = \delta^{ij}\delta(\mathbf{x}) - \delta^{ij}_{\mathrm{tr}}(\mathbf{x})$ can be used to project out the transverse and longitudinal parts of a function $\mathbf{F}(\mathbf{x})$.

(2) **Free propagator as a Green's function** (2 Punkte)
Demonstrate that the free complex scalar propagator is a Green's function of the Klein-Gordon equation:

$$(\partial_\mu \partial^\mu + m^2)\Delta(x, y) = -i\delta(x - y)\,,$$

where $\Delta(x, y) = \langle 0|\mathrm{T}\phi(x)\phi^\dagger(y)|0\rangle$. For simplicity, assume the case of $(1+1)$ dimensions (i.e., one spatial dimension).

(3) **Free propagator and momentum-space action** (3 Punkte)
Consider a scalar field theory defined by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \, .$$

By considering the Fourier transform of the field, show that the action may be written as

$$S = \frac{1}{2}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\tilde{\phi}(-p)(p^2 - m^2)\tilde{\phi}(p)$$

where $p^2 := p_\mu p^\mu$. This provides us with an alternative method for identifying the free propagator $\tilde{G}_0(p)$ as ($i/2$ times) the inverse of the quadratic term in the momentum-space action.

(4) **Kramers degeneracy** (3 Bonus-Punkte)
Study the degeneracies in the spectrum of three coupled spins 1/2,

$$H = J\sum_i \mathbf{S}_i \cdot \mathbf{S}_{i+1}$$

both for open and for periodic boundary conditions and for ferromagnetic ($J < 0$) and antiferromagnetic ($J > 0$) coupling. Use a computer if necessary.
Hint: $\mathbf{S}_i \cdot \mathbf{S}_{i+1} = S_i^z S_{i+1}^z + \frac{1}{2}(S_i^+ S_{i+1}^- + h.c.)$