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Advanced quantum mechanics and quantum field theory, FS 2021

Blatt 1

Submission: 11.03.2021, 12:00H, **on adam in the appropriate folder.**
One file per submission please; the filename HAS TO contain your name, or the submission will not be corrected!
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(1) **Variational derivative II** (1 Punkt)

Show that if $Z_0[J]$ is given by

$$Z_0[J] = \exp\left(-\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y J(x)\Delta(x-y)J(y)\right) ,$$

where $\Delta(x) = \Delta(-x)$, then

$$\frac{\delta Z_0[J]}{\delta J(z_1)} = -\left[\int \mathrm{d}^4y\Delta(z_1-y)J(y)\right] Z_0[J] .$$

(2) **Second-quantized operators** (4 Punkte)

(a) Calculate the Fourier transform of the density operator,

$$\hat{n}_{\mathbf{q}\sigma} = \int \mathrm{d}^3x\ \hat{n}_\sigma(\mathbf{x})e^{-i\mathbf{qx}} = \int \mathrm{d}^3x\ \Psi^\dagger_\sigma(\mathbf{x})\Psi_\sigma(\mathbf{x})e^{-i\mathbf{qx}} .$$

Explain the difference between $\hat{n}_{\mathbf{q}}$ and the occupation-number operator in momentum space, $c^\dagger_{\mathbf{q}\sigma}c_{\mathbf{q}\sigma}$. Does it matter whether the particles are bosons or fermions?

(b) The current density operator in position representation reads

$$\mathbf{j}(\mathbf{r}) = \frac{-ie\hbar}{2m}\sum_{\sigma=\pm\frac{1}{2}}\left[\Psi^\dagger_\sigma(\mathbf{r})\nabla\Psi_\sigma(\mathbf{r}) - \{\nabla\Psi^\dagger_\sigma(\mathbf{r})\}\Psi_\sigma(\mathbf{r})\right]$$

Express its Fourier transform $\mathbf{j}(\mathbf{k}) = \int \mathrm{d}^3r\ \mathbf{j}(\mathbf{r})e^{-i\mathbf{kr}}$ using the $c_{\mathbf{k}\sigma}$, $c^\dagger_{\mathbf{k}\sigma}$.

(c) Non-interacting electrons in a magnetic field are described by the Hamiltonian

$$H = \sum_{i,\sigma}\frac{1}{2m}(\mathbf{p}_i - e\mathbf{A}(\mathbf{x}_i))^2 .$$

Assume that the vector potential fulfills the Coulomb gauge. Write H in second-quantized notation, both in the position and in the momentum representation. Neglect the term $\mathbf{A}^2$.

(3) **2-site fermionic Hubbard model** (5 Punkte + 2 Bonuspunkte)

We consider a toy model of two electrons on two lattice sites L, R, interacting with a short-range interaction. The model Hamiltonian is

$$H = -t \sum_{\sigma=\uparrow,\downarrow} \left(c_{L\sigma}^\dagger c_{R\sigma} + c_{R\sigma}^\dagger c_{L\sigma} \right) + U \left(n_{L\uparrow} n_{L\downarrow} + n_{R\uparrow} n_{R\downarrow} \right)$$

where $t > 0$. Here $c_{L\sigma}^\dagger$ ($c_{L\sigma}$) creates (annihilates) an electron in the σ state on the left site, $c_{R\sigma}^\dagger$ ($c_{R\sigma}$) creates (annihilates) an electron in the σ state on the right site, and $n_{L\sigma} = c_{L\sigma}^\dagger c_{L\sigma}$ ($n_{R\sigma}$) is the particle number operator of left (right) site, which counts the particle number in the σ state.

(a) Interpret the terms of the Hamiltonian.

(b) The dimension of the Hilbert space is six. Give a set of basis vectors of the system and calculate the Hamiltonian matrix.
Hint: make sure you get the correct signs for the matrix elements of the hopping term (fermionic creation and annihilation operators!).

(c) Determine the ground-state energy and the ground state for $U > 0$ (repulsive interaction) at arbitrary ratio $\xi = t/U$. Discuss the limits $\xi \ll 1$ and $\xi \gg 1$.

(d) Consider now the case $U < 0$, i.e., an attractive interaction between the electrons [e.g., mediated by phonons]. Determine the ground-state energy and the ground state and discuss the limits $|\xi| \ll 1$ and $|\xi| \gg 1$, $\xi < 0$.

(e) (2 bonus points) Use a computer to solve the same problem if the on-site interaction is different on the two sites, i.e., $U_L \neq U_R$. Compare with the analytical results obtained in (c) and (d) and study interesting limiting cases.