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Advanced quantum mechanics and quantum field theory, FS 2021

Blatt 0

Besprechung: 05.03.2021; 13:15 – 15:00
(to be discussed in the first exercise class; no submission!)
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The purpose of problems 1 and 2 is to train the use of the formalism of second quantization. The anticommutator/commutator relations for fermionic/bosonic operators read

	momentum representation	position representation
fermions	$\{c_{\mathbf{k}'\sigma'}, c^\dagger_{\mathbf{k}\sigma}\} = \delta_{\mathbf{k}'\mathbf{k}}\delta_{\sigma'\sigma}$	$\{\Psi_{\sigma'}(\mathbf{r}'), \Psi^\dagger_\sigma(\mathbf{r})\} = \delta(\mathbf{r}' - \mathbf{r})\delta_{\sigma'\sigma}$
bosons (spinless)	$[a_{\mathbf{k}'}, a^\dagger_{\mathbf{k}}] = \delta_{\mathbf{k}'\mathbf{k}}$	$[\Phi(\mathbf{r}'), \Phi^\dagger(\mathbf{r})] = \delta(\mathbf{r}' - \mathbf{r})$

All others (like $\{c_{\mathbf{k}'\sigma'}, c_{\mathbf{k}\sigma}\}$, $[a^\dagger_{\mathbf{k}'}, a^\dagger_{\mathbf{k}}]$, $\cdots$) vanish.

(1) **Position and momentum representation**
For free electrons, the relation between position and momentum representation reads

$$\Psi_\sigma(\mathbf{r}) = V^{-\frac{1}{2}} \sum_{\mathbf{k}} c_{\mathbf{k}\sigma} e^{i\mathbf{k}\mathbf{r}}.$$

Write the Hamiltonian

$$H = \sum_{\sigma=\pm\frac{1}{2}} \int \mathrm{d}^3 r \; \Psi^\dagger_\sigma(\mathbf{r}) \big[\frac{-\hbar^2}{2m}\nabla^2 + U(\mathbf{r})\big] \Psi_\sigma(\mathbf{r})$$

in terms of the $c_{\mathbf{k}\sigma}$, $c^\dagger_{\mathbf{k}\sigma}$.

(2) **Tight-binding model in second quantization**
A major part of solid-state physics deals with electrons in a periodic potential. As a simplified model we consider fermionic particles moving on a cubic lattice (lattice constant a). The kinetic energy is assumed to have tight-binding form

$$H = -t \sum_{<i,j>\sigma} \left[c^\dagger_{i\sigma} c_{j\sigma} + c^\dagger_{j\sigma} c_{i\sigma} \right] \,,$$

here, $\sum_{<i,j>}$ is the sum over all nearest neighbors (such that each bond appears only once) and $\sum_\sigma$ is the sum over the two spin directions.

(a) Determine the band structure $\epsilon(\mathbf{k})$ for a d-dimensional cubic lattice ($d = 1, 2, 3$).

(b) Draw the contours $\epsilon(\mathbf{k}) = \text{const.}$ in the (k_x, k_y)-plane for $d = 2$.

Hint: Diagonalize the Hamiltonian by a Fourier transform, $c_{j\sigma} = \frac{1}{\sqrt{N}} \sum_{\mathbf{k}} \exp(i\mathbf{k}\mathbf{r}_j) c_{\mathbf{k}\sigma}$, here, $\mathbf{r}_j$ are the coordinates of the lattice sites; N is their total number.

(3) **Variational derivative I**

(a) Consider the functional $G[f] = \int g(y, f)\mathrm{d}y$. Show that

$$\frac{\delta G[f]}{\delta f(x)} = \frac{\partial g(x, f)}{\partial f} \, .$$

(b) Now consider the functional $H[f] = \int g(y, f, f')\mathrm{d}y$ and show that

$$\frac{\delta H[f]}{\delta f(x)} = \frac{\partial g}{\partial f} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial g}{\partial f'} \, ,$$

where $f' = \partial f / \partial y$.

(c) For the functional $J[f] = \int g(y, f, f', f'')\mathrm{d}y$ show that

$$\frac{\delta J[f]}{\delta f(x)} = \frac{\partial g}{\partial f} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial g}{\partial f'} + \frac{\mathrm{d}^2}{\mathrm{d}x^2} \frac{\partial g}{\partial f''} \, ,$$

where $f'' = \partial^2 f / \partial y^2$.

(4) **Three-dimensional elastic medium**

For a three-dimensional elastic medium, the potential energy is

$$V = \frac{\mathcal{T}}{2} \int \mathrm{d}^3x (\boldsymbol{\nabla}\psi)^2 \, ,$$

and the kinetic energy is

$$T = \frac{\rho}{2} \int \mathrm{d}^3x (\frac{\partial \psi}{\partial t})^2 \, .$$

Here, $\psi(\mathbf{x}, t)$ describes the displacement of the elastic medium at $\mathbf{x}$ at time t, ρ is the mass density, and $\mathcal{T}$ the tension.

Write down the Lagrange density, the action, the Euler-Lagrange equation, and the resulting equation of motion for $\psi(\mathbf{x}, t)$.