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Elektrodynamik, Frühjahrsemester 2019

Blatt 4

Abgabe: 26.3.19, 12:00H (Treppenhaus 4. Stock)
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(1) **Capacitance coefficients** (3 Punkte)

Consider N metallic conductors; the i^{th} conductor carries the total charge Q_i, and its potential is ϕ_i (conductors are equipotentials). Since the potential depends linearly on the charge density and vice versa, there is a linear relation between the ϕ_i and the Q_j:

$$Q_i = \sum_{j=1}^{N} C_{ij}\phi_j; \qquad \phi_i = \sum_{j=1}^{N} [C^{-1}]_{ij} Q_j.$$

The C_{ij} are called *capacitance coefficients*.

(a) Express the potential energy W of this system in terms of the ϕ_i alone and in terms of the Q_i alone.

(b) Show that C_{ij} is symmetric.

(c) Give a physical argument why C_{ij} has to be positive definite. Conclude that the diagonal elements of C and of C^{-1} are positive.

(d) Show that $C_{ij} < 0$ for $i \neq j$.
Hint: Consider e.g. the case $\phi_1 > 0$ and $\phi_i = 0$ for $i > 1$ and take into account that ϕ is harmonic between the conductors.

(2) **Multipole moments of a "physical dipole"** (3 Punkte)

Consider a "physical dipole" consisting of a charge q at $\mathbf{x} = (0, 0, d/2)$ and a charge $-q$ at $\mathbf{x} = (0, 0, -d/2)$.

(a) Write down an expression for the charge density in spherical coordinates.
Hint: The delta-function in spherical coordinates reads
$\delta(\mathbf{x} - \mathbf{x}') = \frac{1}{r^2}\delta(r - r')\delta(\varphi - \varphi')\delta(\cos\theta - \cos\theta')$.

(b) Calculate the spherical multipole moments $q_{lm} = \int \mathrm{d}^3x'\, Y^*_{lm}(\theta', \varphi') r'^l \rho(\mathbf{x}')$ for this charge density.
Hint: $Y_{lm}(\theta = 0, \varphi) = \delta_{m0}\sqrt{\frac{2l+1}{4\pi}}; \qquad Y_{lm}(\pi - \theta, \varphi + \pi) = (-1)^l Y_{lm}(\theta, \varphi)$.

(c) Interpret the first two non-vanishing terms of the spherical multipole expansion for the potential. Use a computer to compare with the exact potential.

(d) How is it possible that a "dipole" has higher multipole moments?
Can you construct a charge distribution that has a dipole moment but no higher multipole moments?

(3) **Dielectric cylinder** (4 Punkte + 3 Bonus-Punkte)

A very long, right circular, cylindrical shell of dielectric constant $\epsilon_r = \epsilon/\epsilon_0$ and inner and outer radii a and b, respectively, is placed in a previously uniform electric field $\mathbf{E} = E_0\,\mathbf{e}_x$ in x-direction with its axis perpendicular to the field (say, in z-direction). The medium inside and outside the cylinder has a dielectric constant of unity.

The general solution of the Laplace equation in two dimensions in polar coordinates reads

$$\begin{aligned}\phi(s,\varphi) = A_0 + C_0 \ln s + \sum_{m>0} s^m [A_m \cos(m\varphi) + B_m \sin(m\varphi)] \\ + \sum_{m>0} s^{-m} [C_m \cos(m\varphi) + D_m \sin(m\varphi)]\,. \end{aligned} \tag{1}$$

Either take this for granted, or prove it (not hard) and get 3 bonus points!

(a) Write down the appropriate Ansatz for the potential ϕ in the three regions $s < a$, $a \leq s < b$, and $b \leq s$, neglecting end effects.
Hint: use physical arguments to show that $A_0 = C_0 = B_m = D_m = 0$ in Eq. (1).
Hence, $\phi(s,\varphi) = \sum_{m>0} (A_m s^m + C_m s^{-m}) \cos(m\varphi)$.

(b) Write down the boundary conditions on the fields $\mathbf{E}$ and $\mathbf{D}$. Use them to determine the potential in the three regions.
Hint: The boundary conditions result in a system of linear equations for the coefficients A_m, C_m. Show without further calculation that for $m > 1$, there is only the trivial solution $A_m = C_m = 0$.

(c) Calculate and discuss the fields $\mathbf{E}$ and $\mathbf{D}$ in the three regions. Consider the limits $\epsilon_r \to 1$, $\epsilon_r \to \infty$, and $a \to b$. Discuss the limiting forms of your solution appropriate for a solid dielectric cylinder in a uniform field, and a cylindrical cavity in a uniform dielectric.

(d) Illustrate the fields $\mathbf{E}$ and $\mathbf{D}$ numerically for $b = 2a$ and $\epsilon_r = 5$.