Departement Physik, Universität Basel
Prof. C. Bruder (Zimmer 4.2, Tel.: 207 36 92, Christoph.Bruder@unibas.ch)

Elektrodynamik, Frühjahrsemester 2019

Blatt 11 (Zusatzblatt)

Besprechung: **nach Vereinbarung**

(1) **Lorentz invariants** (5 Punkte)

(a) Explain in your own words why $F_{\alpha\beta}F^{\alpha\beta}$ is Lorentz invariant. Here, $F^{\alpha\beta}$ is the electromagnetic field tensor. Conclude that $\mathbf{E}^2 - c^2\mathbf{B}^2$ is Lorentz invariant.

(b) Consider the 4-dimensional Levi-Civita tensor $\epsilon^{\alpha\beta\gamma\delta}$ that is $+1(-1)$ if $(\alpha\beta\gamma\delta)$ is an even (odd) permutation of $(0,1,2,3)$, and 0 otherwise. Show that ϵ transforms as a 4-tensor, i.e. that

$$\epsilon'^{\;\alpha\beta\gamma\delta} = \Lambda^{\alpha}{}_{\alpha'}\Lambda^{\beta}{}_{\beta'}\Lambda^{\gamma}{}_{\gamma'}\Lambda^{\delta}{}_{\delta'}\,\epsilon^{\alpha'\beta'\gamma'\delta'}\;,$$

has the same combinatorial properties as $\epsilon^{\alpha\beta\gamma\delta}$. Consider only proper Lorentz transformations with $\det\Lambda = 1$.

(c) Explain in your own words why $\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}$ is Lorentz invariant. Conclude that $\mathbf{E}\cdot\mathbf{B}$ is Lorentz invariant.

(d) Show that $\epsilon_{\alpha\beta\gamma\delta} = -\epsilon^{\alpha\beta\gamma\delta}$.

(2) **Energy-momentum tensor** (5 Punkte)

The energy-momentum tensor is defined as

$$T^{\alpha\beta} = \frac{1}{\mu_0}\left(g^{\alpha\mu}F_{\mu\lambda}F^{\lambda\beta} + \frac{1}{4}g^{\alpha\beta}F_{\mu\lambda}F^{\mu\lambda}\right)\;,$$

here, $F^{\alpha\beta}$ is electromagnetic field tensor.

(a) Express the different components of $T^{\alpha\beta}$ in terms of $\mathbf{E}$, $\mathbf{B}$ and identify them.

(b) Show that $T^{\alpha\beta}$ fulfills

$$\partial_\alpha T^{\alpha\beta} = -F^{\beta\lambda}j_\lambda\;, \tag{1}$$

here, j^α is the current density 4-vector.

(c) Express the different components of Eq. (1) in terms of $\mathbf{E}$, $\mathbf{B}$, ρ and $\mathbf{j}$ and interpret them.

(3) **Principle value / Hilbert transform** (5 Punkte)

Show that for $z \in \mathbb{R}$,

$$\mathrm{P}\int_{-\infty}^{\infty} \frac{\cos t}{t-z}\,\mathrm{d}t = -\pi \sin z\,.$$

Similarly, show that

$$\mathrm{P}\int_{-\infty}^{\infty} \frac{\sin t}{t-z}\,\mathrm{d}t = \pi \cos z\,.$$

Here, $\mathrm{P}\int$ denotes the principle value of the integral.

(4) **Kramers-Kronig relations** (5 Punkte)

In the lecture, we discussed that causality guarantees the dielectric function $\epsilon(\omega)$ to be analytical in the upper half of the complex ω-plane. From Cauchy's theorem, we can then derive the so-called Kramers-Kronig relations between the real and imaginary parts of $\epsilon(\omega)$. Here, P denotes the principal part of the integral.

$$\begin{aligned} \mathrm{Re}\,\epsilon(\omega)/\epsilon_0 &= 1 + \frac{2}{\pi}\mathrm{P}\int_0^{\infty} \frac{\omega'\mathrm{Im}\,\epsilon(\omega')/\epsilon_0}{\omega'^2-\omega^2}\,\mathrm{d}\omega'\,, \\ \mathrm{Im}\,\epsilon(\omega)/\epsilon_0 &= -\frac{2\omega}{\pi}\mathrm{P}\int_0^{\infty} \frac{[\mathrm{Re}\,\epsilon(\omega')/\epsilon_0 - 1]}{\omega'^2-\omega^2}\,\mathrm{d}\omega'\,. \end{aligned} \tag{2}$$

Use the Kramers-Kronig relations to calculate the real part of $\epsilon(\omega)$, given the imaginary part of $\epsilon(\omega)$ for positive ω as

(a) $\mathrm{Im}\,\epsilon(\omega)/\epsilon_0 = \lambda[\Theta(\omega-\omega_1) - \Theta(\omega-\omega_2)]$, $\omega_2 > \omega_1 > 0$.

(b) (hard!) $\mathrm{Im}\,\epsilon(\omega)/\epsilon_0 = \dfrac{\lambda\gamma\omega}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$.

In each case sketch the behavior of $\mathrm{Im}\,\epsilon(\omega)$ and the result for $\mathrm{Re}\,\epsilon(\omega)$ as functions of ω.

(5) **Kramers-Kronig relations for a conductor** (5 Punkte)

The oscillator model can be extended to metals by assuming that there are f_0 "free" electrons per molecule ($\omega_0 = 0$). This leads to an additional term

$$i\frac{Ne^2 f_0}{m\omega(\gamma_0 - i\omega)}$$

in $\epsilon(\omega)$.

(a) Show that by using Ohm's law $\mathbf{j} = \sigma\mathbf{E}$ on the right-hand side of the 4^{th} Maxwell equation and attributing the "bound-electron" part of ϵ to $\mathbf{D}$, the additional term can be written as $\dfrac{i\sigma}{\omega}$.

(b) Discuss the extension of the Kramers-Kronig relations for a medium with a static electrical conductivity σ. Show that the first equation in (2) is unchanged, but that the second is changed into

$$\mathrm{Im}\,\epsilon(\omega)/\epsilon_0 = \frac{\sigma}{\omega} - \frac{2\omega}{\pi}\mathrm{P}\int_0^{\infty} \frac{[\mathrm{Re}\,\epsilon(\omega')/\epsilon_0 - 1]}{\omega'^2-\omega^2}\,\mathrm{d}\omega'$$

[Hint: Consider $\epsilon(\omega) - i\sigma/\omega$ as analytic for $\mathrm{Im}\,\omega > 0$.]

(6) **General properties of $\epsilon(\omega)$** (5 Punkte)

(a) Use $\epsilon(-\omega) = \epsilon^*(\omega^*)$ and the analyticity of $\epsilon(\omega)$ for $\operatorname{Im}\omega \geq 0$ to prove that on the positive imaginary axis $\epsilon(\omega)/\epsilon_0$ is real and monotonically decreasing away from the origin toward unity as $\omega \to i\infty$. (Remember that $\operatorname{Im}\epsilon \geq 0$ for real positive frequencies since this quantity describes absorption).
Hint: $\operatorname{Re}[\epsilon(\omega)/\epsilon_0 - 1] \sim \omega^{-2}$; $\operatorname{Im}[\epsilon(\omega)/\epsilon_0] \sim \omega^{-3}$ for $\omega \to \infty$.

(b) With the assumption that $\operatorname{Im}\epsilon$ vanishes for finite real ω only at $\omega = 0$, show that $\epsilon(\omega)$ has no zeros in the upper half-ω-plane.

(c) Write down a Kramers-Kronig relation for $\epsilon_0/\epsilon(\omega)$ and deduce a sum rule similar to the one derived in class, but as an integral over $\operatorname{Im}[\epsilon_0/\epsilon(\omega)]$.

(d) Determine $\operatorname{Im}[\epsilon(\omega)/\epsilon_0]$ and $\operatorname{Im}[\epsilon_0/\epsilon(\omega)]$ using the one-resonance oscillator model for $\epsilon(\omega)$. Verify explicitly that the sum rules derived in the lecture and in (c) are satisfied.