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Elektrodynamik, Frühjahrsemester 2019

Blatt 1

Abgabe: 26.2.19, 12:00H (Treppenhaus 4. Stock)
Tutor: Frank Schäfer, Zi: 4.13

Die **Übungskreditpunkte** erhält, wer sowohl 50% der Punkte aus den Hausaufgaben erreicht als auch 50% der Punkte aus dem schriftlichen Test am Ende des Semesters.

(1) **ϵ-symbol** (4 Punkte)

The ϵ-symbol (or Levi-Civita tensor) in 3 dimensions is defined by

$$\epsilon_{ijk} := \begin{cases} 1, & \text{if } (i,j,k) \text{ can be obtained from (1,2,3) by an } \textit{even} \text{ permutation,} \\ -1, & \text{if } (i,j,k) \text{ can be obtained from (1,2,3) by an } \textit{odd} \text{ permutation,} \\ 0, & \text{otherwise.} \end{cases}$$

A permutation is called even (odd) if it can be expressed as an even (odd) number of pairwise exchanges.

(a) Prove the following identity between the ϵ-symbol and the basis vectors $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ of a 3-dimensional Cartesian coordinate system:

$$\epsilon_{ijk} = \vec{e}_i\,(\vec{e}_j \times \vec{e}_k)\,. \tag{1}$$

(b) Prove the following identities [e.g., by using Eq. (1)]:

i. $\vec{c} = \vec{a} \times \vec{b} \Leftrightarrow c_i = \sum_{jk} \epsilon_{ijk} a_j b_k$.
Conclude that $(\vec{\nabla} \times \vec{v})_i = \sum_{jk} \epsilon_{ijk} \partial_j v_k$, here, $\partial_j := \dfrac{\partial}{\partial x_j}$

ii. $\sum_k \epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

iii. $\sum_{jk} \epsilon_{ijk}\epsilon_{ljk} = 2\delta_{il}$

iv. $\sum_{ijk} \epsilon_{ijk}\epsilon_{ijk} = 6$

Formulas of this type can be written more elegantly by using Einstein's summation convention: whenever an index variable appears twice in a product, summation over all of its possible values is implied: e.g., $a_i b_i := \sum_{i=1}^{3} a_i b_i$.

Hence $\vec{\nabla} \cdot \vec{A} = \partial_j A_j$, or $(\vec{\nabla} \times \vec{A})_i = \epsilon_{ijk} \partial_j A_k$.

(2) **Important identities with the ∇ operator** (3 Punkte)

Prove the following identities by using the properties of the ϵ-symbol ϵ_{ijk} given in problem 2:

(a) $\vec{\nabla} \times (\vec{\nabla}\varphi) = 0$

(b) $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$

(c) $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$

(d) $\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B})$

where $\varphi(\vec{r})$ is a scalar field, and $\vec{A}(\vec{r})$ and $\vec{B}(\vec{r})$ are vector fields.

(3) **Electrostatics** (3 Punkte)

(a) Show that the divergence of $\dfrac{\vec{x}}{|\vec{x}|^3}$ vanishes for $\vec{x} \neq 0$.

(b) Using Gauss's theorem, show that the volume integral of $\dfrac{\vec{x}}{|\vec{x}|^3}$ over a sphere centered at the origin is 4π.

(c) Conclude that $\nabla_x^2 (\dfrac{1}{|\vec{x} - \vec{x}\,'|}) = -4\pi\delta(\vec{x} - \vec{x}\,')$.

(d) We consider a time-independent charge density $\rho(\vec{x})$. Show that the scalar potential is given by

$$\phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int d^3x' \frac{\rho(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|} ,$$

and the electric field by

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int d^3x' \rho(\vec{x}\,') \frac{\vec{x} - \vec{x}\,'}{|\vec{x} - \vec{x}\,'|^3} .$$

(4) **Relativistic Doppler effect** (2 Extra-Punkte)

A source emits plane monochromatic electromagnetic waves (frequency ω) in z-direction. An observer moves with velocity v in x- resp. in z-direction. Calculate the frequency ω' that she measures for the wave, and the direction in which the wave propagates. Plot and discuss the frequency as a function of v.

Hint: $(\omega/c, \mathbf{k})$ is a 4-vector, i.e., has the same transformation properties as $(ct, \mathbf{x})$.